The deflection of light ray in strong field: a material medium approach

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The amount of deflection for a ray of light passing close to a gravitational mass can be worked out from the null geodesic the ray follows [MTW 1972,Weinberg 1972; Schneider, Ehlers, Falco 1999].

Such expressions involving Elliptical Integrals were first given by Darwin in 1959.

The calculation of higher order deflection terms, from the null geodesic, has been performed by Iyer & Petters 2007; Virbhadra & Ellis 2000; Frittelli, Killing & Newman 2000.

More recently, for strong gravitational field, lensing calculations have been done by Bisnovatyi-Kogan & Tsupko 2008.

The deflection of a light ray can be alternately calculated by considering the light ray to be passing through a *material medium*, due to the effect of gravitation.

This concept of *equivalent material medium* was discussed by Balazs as early as in 1958, to calculate the effect of a rotating body on a light ray.

Plebanski had also utilized this concept in 1960, to study the scattering of a plane electromagnetic wave by gravitational field. Plebanski also mentioned that this concept of *equivalent material medium* was first pointed out by Tamm in 1924.

Atkinson (1965) investigated the allowed trajectories of light rays near a massive star and obtained an expression for velocity of light at an arbitrary point.

A general procedure for utilizing this concept, for deflection calculation was worked out by Felice (1971).

Later this concept was also used by Mashoon (1973,1975), to calculate the deflection and polarization due to the Schwarzschild and Kerr black holes.

Fischbach and Freeman (1980), derived the effective refractive index of the medium and calculated the second order contribution to the gravitational deflection.

Nandi & Islam (1995) and Evans, Nandi & Islam(1996) derived and used the effective refractive index values to calculate gravitational time delay and trajectories of light rays.

Sereno (2003) had used this idea, for gravitational lensing calculation by using Fermat's principle.

More recently Ye and Lin (2008), emphasized the simplicity of this approach and calculated the gravitational time delay and the effect of lensing.

1 The effective refractive index

The Schwarzschild equation (1961) (with $r_g = \frac{2GM}{c^2}$):

$$ds^2 = (1 - \frac{r_g}{r})c^2dt^2 - r^2(sin^2\theta d\phi^2 + d\theta^2) - \frac{dr^2}{(1 - \frac{r_g}{r})} \quad (1)$$

Written in an isotropic form (Landau & Lifshitz 1980):

$$ds^2 = (\frac{1 - r_g/(4\rho)}{1 + r_g/(4\rho)})^2 c^2dt^2 - (1 + \frac{r_g}{4\rho})^4(d\rho^2 + \rho^2(sin^2\theta d\phi^2 + d\theta^2)) \quad (2)$$

where

$$\rho = \frac{1}{2}[(r - \frac{r_g}{2}) + r^{1/2}(r - r_g)^{1/2}] \quad (3)$$

The quantity $(d\rho^2 + \rho^2(sin^2\theta d\phi^2 + d\theta^2))$ has the dimension of square of infinitesimal length vector $d\overrightarrow{\rho}$.

Setting $ds = 0$, we get the velocity of light :

$$v(\rho) = \frac{(1 - \frac{r_g}{4\rho})c}{(1 + \frac{r_g}{4\rho})^3} \quad (4)$$

Now

$$\begin{aligned} v(r) = \quad & v(\rho)\frac{dr}{d\rho} \\ & = v(\rho)[(1 + \frac{r_g}{4\rho})^2 - \frac{r_g}{2\rho}(1 + \frac{r_g}{4\rho})] \\ & = (\frac{r_g - 4\rho}{r_g + 4\rho})^2 c \end{aligned} \quad (5)$$

Substituting the value of ρ from Eqn (2) in Eqn.(6), we get:

$$\begin{aligned} v(r) = & \quad (\frac{r_g/2 - 2\rho}{r_g/2 + 2\rho})^2 c \\ = & \quad (\frac{r_g/2 - ((r - \frac{r_g}{2}) + r^{1/2}(r - r_g)^{1/2})}{r_g/2 + ((r - \frac{r_g}{2}) + r^{1/2}(r - r_g)^{1/2})})^2 c \\ = & \quad (\frac{r_g - r - r^{1/2}(r - r_g)^{1/2}}{r + r^{1/2}(r - r_g)^{1/2}})^2 c \\ = & \quad \frac{c(r - r_g)}{r} \end{aligned} \tag{6}$$

Therefore the refractive index $n(r)$:

$$n(r) = \frac{c}{v(r)} = \frac{r}{r - r_g} \tag{7}$$

Here we note that, the values of refractive index derived following either Atkinson (1965) or Fischback & Freeman (1980) lead to an expression containing terms of some infinite converging series.

Fischback & Freeman (1980) used an infinite convergent series (1+ A/r + B/r^2+...) for refractive index $n(r)$, where $A = r_g$ (Schwarzschild radius) and B is some function of r_g.

Fischback and Freeman (1980) estimated light deflection by a massive object by truncating the series at some stage, whereas no deflection values were calculated with Atkinson's refractive index expressions.

2 The trajectory of a light ray

The light ray and the gravitational mass together define a plane.

The equation of a ray in (r, θ) plane can be written as (Born & Wolf 1947) :

$$\theta = A. \int_{r_{\odot}}^{\infty} \frac{dr}{r\sqrt{n^2 r^2 - A^2}} \tag{8}$$

The trajectory is such that $n(r).d$ always remains a *constant*, where d is the perpendicular distance between the trajectory of the light ray from the origin (gravitational mass) and the *constant* is taken here as A.

Here light is approaching from asymptotic infinity ($r = -\infty$) to the gravitational mass. The closest distance of approach, for the approaching ray is b and the ray goes to $r = \infty$, after undergoing certain amount of deflection ($\triangle\phi$), by the presence of gravitational mass.

The parameter b can be replaced by solar radius $r_{\odot}$. When the light ray passes through the closest distance of approach (ie $r = b$ or $r_{\odot}$), the tangent to the trajectory becomes perpendicular to the vector $\overrightarrow{r}$ (which is $\overrightarrow{r}_{\odot}$). Therefore, we can write $A = n(r_{\odot})r_{\odot}$.

With this geometry, the value of deflection ($\triangle\phi$), can be written as Ye and Lin (2008):

$$\triangle\phi = 2 \int_{r_{\odot}}^{\infty} \frac{dr}{r\sqrt{(\frac{n(r).r}{n(r_{\odot}).r_{\odot}})^2 - 1}} - \pi \tag{9}$$

We denote the above integral in Eqn. (9) by I and write

$$
\begin{aligned}
I = \quad & \int_{r_\odot}^{\infty} \frac{dr}{r\sqrt{(\frac{n(r).r}{n(r_\odot).r_\odot})^2 - 1}} \\
& = n(r_\odot)r_\odot \int_{r_\odot}^{\infty} \frac{dr}{r\sqrt{(n(r).r)^2 - (n(r_\odot).r_\odot)^2}} \\
& = n(r_\odot)r_\odot \int_{r_\odot}^{\infty} \frac{dr}{r^2\sqrt{\frac{1}{(1-\frac{r_g}{r})^2} - \frac{r_\odot^2 r^{-2}}{(1-\frac{r_g}{r_\odot})^2}}}
\end{aligned}
\tag{10}
$$

Change the variable to $x = \frac{r_g}{r}$ and introduce $a = \frac{r_g}{r_\odot}$:

$$
\begin{aligned}
I = \quad & n_\odot r_\odot \int_{a}^{0} \frac{-x^{-2} r_g dx}{r^2\sqrt{\frac{1}{(1-x)^2} - \frac{x^2}{(a(1-a))^2}}} \\
& = n_\odot r_\odot \int_{a}^{0} \frac{-x^{-2} r_g dx}{xr^2\sqrt{\frac{1}{(x(1-x))^2} - \frac{1}{(a(1-a))^2}}} \\
& = \frac{n_\odot r_\odot}{r_g} \int_{0}^{a} \frac{dx}{x\sqrt{\frac{1}{(x(1-x))^2} - \frac{1}{(a(1-a))^2}}} \\
& = \frac{n_\odot r_\odot}{r_g} \int_{0}^{a} \frac{(1-x)dx}{\sqrt{1 - \frac{(x(1-x))^2}{(a(1-a))^2}}}
\end{aligned}
\tag{11}
$$

We denote the quantity $1/(a(1-a))$ by D. $\Longrightarrow$

$$D = \frac{r_\odot^2}{r_g(r_\odot - r_g)} \tag{12}$$

We split the above Integral, as a sum of two Integrals:

$$\begin{aligned} I = \quad & (\frac{n_\odot r_\odot}{r_g})[\int_0^a \frac{(1-2x)dx}{\sqrt{1-D^2x^2(1-x)^2}} + \int_0^a \frac{xdx}{\sqrt{1-D^2x^2(1-x)^2}}] \\ &= (\frac{n_\odot r_\odot}{r_g})\int_0^a \frac{(1-2x)dx}{\sqrt{1-D^2x^2(1-x)^2}} + (\frac{n_\odot r_\odot}{r_g})\int_0^a \frac{xdx}{\sqrt{1-D^2x^2(1-x)^2}} \\ &= (\frac{n_\odot r_\odot}{r_g})I_1 + (\frac{n_\odot r_\odot}{r_g})I_2 \end{aligned} \tag{13}$$

where $I_1 = \int_0^a \frac{(1-2x)dx}{\sqrt{1-D^2x^2(1-x)^2}}$ and $I_2 = \int_0^a \frac{xdx}{\sqrt{1-D^2x^2(1-x)^2}}$.

Now

$$\frac{n_\odot r_\odot}{r_g} = \frac{1}{1-a}.\frac{1}{a} = \frac{1}{a(1-a)} = D$$

Changing the variable from x to $y = Dx(1-x)$, $\Longrightarrow D(1-2x)dx = dy$.

The upper and lower limits $x = 0$ and $x = a \Longrightarrow$

$y = 0$ and $y = Da(1 - \frac{r_g}{r_\odot}) = \frac{1}{a(1-a)}a(1-a) = 1$.

Therefore for the first part in Eqn (13) we can write :

$$
\begin{aligned}
(\frac{n_\odot r_\odot}{r_g})I_1 = & \int_0^a \frac{D(1-2x)dx}{\sqrt{1-D^2x^2(1-x)^2}} \\
& = \int_0^1 \frac{dy}{\sqrt{1-y^2}} \\
& = [sin^{-1}y]_0^1 \\
& = \pi/2 \qquad (14)
\end{aligned}
$$

Therefore, from Eqn (9), one may write the amount of deflection as:

$$
\begin{aligned}
\triangle\phi = & 2\int_{r_\odot}^{\infty} \frac{dr}{r\sqrt{(\frac{n(r).r}{n(r_\odot).r_\odot})^2-1}} - \pi \\
& = 2(\frac{n_\odot r_\odot}{r_g})I_1 + 2(\frac{n_\odot r_\odot}{r_g})I_2 - \pi \\
& = \pi + 2(\frac{n_\odot r_\odot}{r_g})I_2 - \pi \\
& = (\frac{2n_\odot r_\odot}{r_g})\int_0^a \frac{xdx}{\sqrt{1-D^2x^2(1-x)^2}} \\
& = 2D\int_0^a \frac{xdx}{\sqrt{1-D^2x^2(1-x)^2}} \qquad (15)
\end{aligned}
$$

One can obtain (using Mathematica etc....)

$$\int \frac{xdx}{\sqrt{1-D^2x^2(1-x)^2}} = 2\frac{(\sqrt{D}+\sqrt{D-4})E-(2\sqrt{D-4})F}{D(\sqrt{D+4}-\sqrt{D-4})} \tag{16}$$

where $E \equiv E(p,q^2)$ is the Elliptic Integral of first kind and $F \equiv F(-q,p,q^2)$ is Incomplete Elliptic Integral of Third kind.

The arguments are :

$$p = \quad \arcsin\sqrt{\frac{(\sqrt{D-4}-\sqrt{D+4})(\sqrt{D-4}+(2x-1)\sqrt{D})}{(\sqrt{D-4}+\sqrt{D+4})(\sqrt{D-4}-(2x-1)\sqrt{D})}} \tag{17}$$

$$q = \quad \frac{(\sqrt{D-4}+\sqrt{D+4})}{(\sqrt{D-4}-\sqrt{D+4})} \tag{18}$$

Finally :

$$\triangle\phi = 4\left\{\frac{(\sqrt{D}+\sqrt{D-4})E-(2\sqrt{D-4})F}{(\sqrt{D+4}-\sqrt{D-4})}\right\}_{x=0}^{x=a} \tag{19}$$

where $D = \frac{r_{\odot}^2}{r_g(r_{\odot}-r_g)}$ and $a = r_g/r_{\odot}$.

For a Sun grazing ray, as a test case :

The closest distance of approach $\cong$ solar radius

$r_{\odot} = 695500$ km

Schwarzschild radius for Sun $r_g = 3$ km.

We get $a = (r_g/r_{\odot}) = 1/231833$ and $D = 231834$.

Finally, using the expression

$$\triangle\phi = 4\left\{\frac{(\sqrt{D}+\sqrt{D-4})E-(2\sqrt{D-4})F}{(\sqrt{D+4}-\sqrt{D-4})}\right\}_{x=0}^{x=a} \tag{20}$$

we get

$\triangle\phi = 8.62690E10^{-6}$ radians or 1.77943 *arc sec.*

3 Comparison with other expressions

Fischback and Freeman had used an infinite convergent series (1+ A/r + B/r^2+...) for refractive index $n(r)$,

where $A = r_g$ (Schwarzschild radius) and B is some function of r_g.

Considering only the first order term in $n(r)$ ie $n(r) = 1 + A/r$, deflection term was calculated as : $2r_g/r_\odot^2$ or $4GM/(c^2 r_\odot)$.

Present work :

the refractive index $n(r) = r/(r - r_g)$ and can be expressed as:

$$n(r) = 1 + a + a^2 + a^3 + ... \tag{21}$$

where $a = r_g/r_\odot$.

So, refractive index in Fischback and Freeman(1980) and in the present work are the same in weak field limit.

Ye and Lin (2008), had also calculated first order deflection term and the refractive index value they used was $n(r) = exp(2GM/(rc^2))$. This expression for refractive index is also same as what has been derived in the present work, for weak field limit(considering terms up to first order). However, these authors did not derive higher order terms for gravitational deflections.

4 Tests under Boundary conditions :

No field condition:

From the present work, it is clear that when one takes $n(r) = 1$, space is flat and there will be no deflection. This is true when we substitute $n(r) = 1$ in Eqn. (9), we get $\triangle\phi = 0$.

Under weak field conditions:

In the present work, the weak field value of refractive index is $n(r) = 1+a$ and under weak field we can also write $1/D = a(1-a) \sim a$ and $x(1-x) \sim x$, as $x, a << 1$. Substituting, these weak field approximations, into the final integral expression (cnf. Eqn (15)) :

$$\begin{aligned}
\triangle\phi = & \quad 2D\int_0^a \frac{xdx}{\sqrt{1-D^2x^2(1-x)^2}} \\
\sim & \quad 2\int_0^a \frac{xdx}{\sqrt{a^2-x^2}} \\
= & \quad 2a \\
= & \quad 2\frac{r_g}{r_\odot} \\
= & \quad \frac{4GM}{c^2 r_\odot}
\end{aligned} \tag{22}$$

This confirms the expression for deflection derived here for strong field coincides to the standard expression under weak field limit.

In the present work, the quantity D was used (in the equations from (14) to (21)) as a substitution for $1/(a(1-a))$. Now, D can be expressed by the converging series :

$$D = 1/a + 1 + a + a^2 + a^3 + ... \tag{23}$$

Therefore, in the weak field limit in order to calculate the value of deflection, we can substitute $D = 1/a$ in Eqn (20), instead of $D = 1/(a(1-a))$. Thus by making the above approximations $D \sim 1/a$ and $a << 1$, into Eqn (20), as and when appropriate, we get:

$$\begin{aligned} \triangle\phi = &\quad 4\left\{\frac{(\sqrt{D}+\sqrt{D-4})E - (2\sqrt{D-4})F}{(\sqrt{D+4}-\sqrt{D-4})}\right\}_{x=0}^{x=a} \\ \sim &\quad (2/a)\left\{E - F\right\}_{x=0}^{x=a} \end{aligned} \tag{24}$$

With the same weak field approximation for the quantities in Eqns. (17) and (18), we obtain $p \sim arcsin(\sqrt{2}a)$ at $x = 0$, $p \sim 0$ at $x = a$ and $q = -1/(2a)$. As a result we get under weak field limit:

$$\triangle\phi = -(2/a)(E(arcsin(\sqrt{2}a), 1/(4a^2)) - F(1/(2a), arcsin(\sqrt{2}a), 1/(4a^2)) \tag{25}$$

where E is the Elliptic Integral of first kind and F is Incomplete Elliptic Integral of Third kind, as discussed earlier. Now by substituting the numerical value of $a = r_g/r_{\odot}$ we can get the weak field deflection value from Eqn (25).

The higher order deflection terms have been also evaluated under strong field, by using methods other than material medium approach.

Darwin(1961), had calculated first order strong field deflection term, using logarithmic series.

Iyer and Petters (2007) had calculated the deflection term in the strong field, with an expression containing complete and incomplete elliptical integrals. From this general expression, the authors could calculate the first order strong deflection term, calculated earlier by Darwin(1961).

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