Selected topics on the functional renormalization group and its applications

Gian Paolo Vacca

INFN - Bologna

Istituto Nazionale di Fisica Nucleare

PAFT2018, Vietri sul mare , 27 Marzo 2018

Outline

- Functional RG approach to QFTs
  - Perturbative
  - Wilsonian (non perturbative)
- Multicritical Yukawa theories
- Applications to Hamiltonian systems:
  - Quantum mechanics
  - RFT for Regge limit of QCD
- Conclusions

Introduction

Physical systems, very different at microscopic level, can show phases characterized by the same Universal behavior when the correlation length diverges (2nd order phase transition).

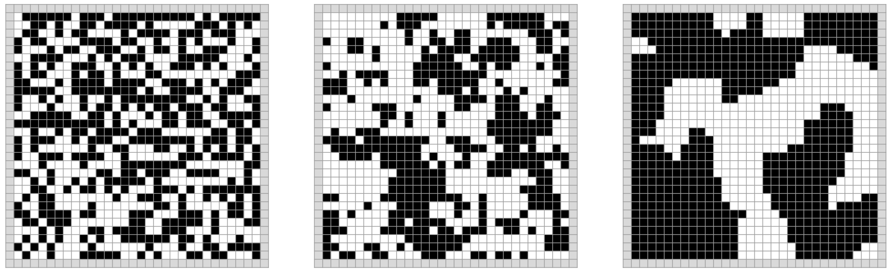
Critical phenomena are conveniently described by Quantum and Statistical Field Theories.

Most famous example:

3D Ising universality class (Magnetic systems, Water) in a Landau-Ginzburg description as a scalar QFT,

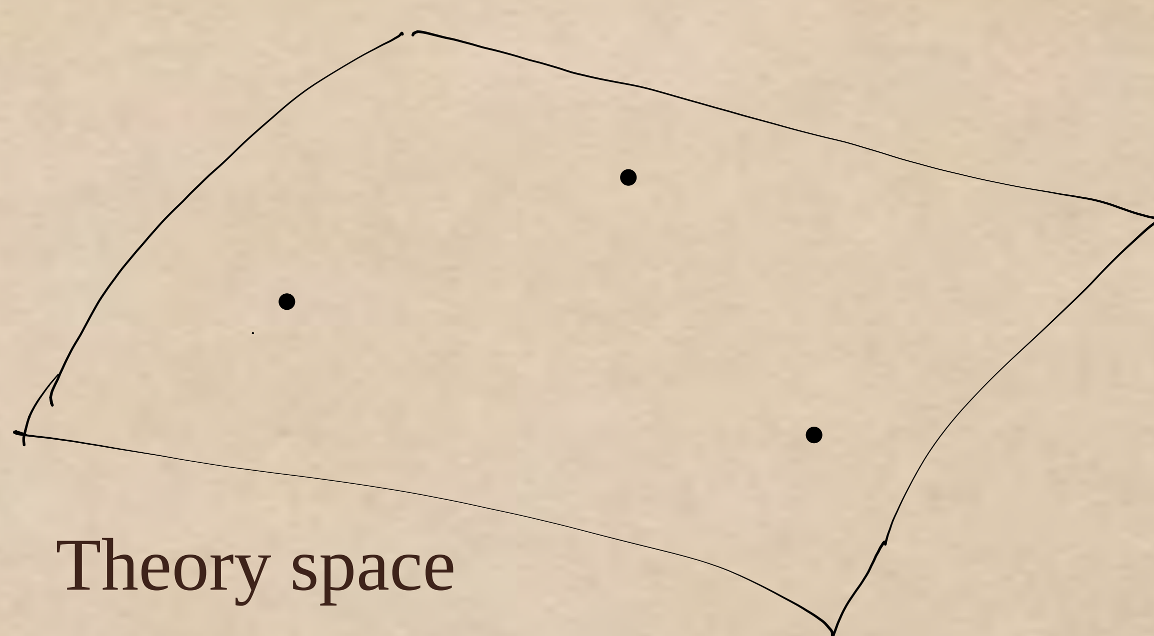
$$S = -J \sum_{\langle ij \rangle} s_i s_j + B \sum_i s_i$$

$$s_i = \pm 1$$



RG is the proper tool to investigate related questions

Critical theories



Theory space
(fields and symmetries)

The critical theories are points in a suitable theory space characterized by scale invariance. If there is Poincare’ invariance it is often lifted to conformal invariance

RG

In a Renormalization Group description critical field theories are associated to fixed points of the flow, where scale invariance is realized.

- These fixed points may control the IR behavior of the theories. (example: Wilson-Fisher fixed point) Wilson (1971), Wilson and Fisher (1972)
- Fundamental physics in a QFT description require renormalizability conditions which in the most general case goes under the name of Asymptotic Safety: existence of a fixed point with a finite number of UV attractive directions. Asymptotic freedom is a particular case with a gaussian fixed point.

Weinberg (1979)

Common formulations:

- Perturbation theory in presence of small parameters, e.g. ε-expansion below the critical dimension
- Wilsonian non perturbative, exact equations but not solvable in practice. (Polchinski and Wetterich/Morris equations)

Action description

The main constraints are given by the field content and the symmetries, but this leaves still too many possible theories for a generic dimension d.

It is therefore useful to start from some kind of Landau-Ginzburg description to single out some possible solutions.

$$S = \int \mathrm{d}^d x \sum_i g_i O_i(\phi)$$

- This is the starting point for an RG analysis.

Couplings are coordinates in theory space, spanned by a basis of operators

- The points corresponding to critical theories may be CFT fixed by the **Conformal data**: the scaling dimensions of the primary operators and the structure constants defining their 3 point correlators. No lagrangian formulation is required.

Universal data and RG

How to get in an RG framework informations on the critical theory?
If conformal, the so called conformal data?

- For example in the perturbative ε-expansion approximation using the universal beta function coefficients, e.g. in a massless $\overline{\text{MS}}$ scheme

Critical quantities are encoded in the expansion coefficients describing the flow around the scale invariant point: $\beta^i(g_*) = 0$

$$\beta^k(g_* + \delta g) = \sum_i M^k{}_i\, \delta g^i + \sum_{i,j} N^k{}_{ij}\, \delta g^i\, \delta g^j + O(\delta g^3)$$

$$M^i{}_j \equiv \left.\frac{\partial \beta^i}{\partial g^j}\right|_* \qquad\qquad N^i{}_{jk} \equiv \frac{1}{2} \left.\frac{\partial^2 \beta^i}{\partial g^j \partial g^k}\right|_*$$

Moving to a diagonal basis in the linear sector

$$\sum_{i,j} \mathcal{S}^a{}_i\, M^i{}_j\, (\mathcal{S}^{-1})^j{}_b = -\theta_a \delta^a{}_b$$

Universal data and RG

$$\theta_a = d - \Delta_a$$

$$\tilde{C}^a{}_{bc} = \sum_{i,j,k} \mathcal{S}^a{}_i \, N^i{}_{jk} \, (\mathcal{S}^{-1})^j{}_b \, (\mathcal{S}^{-1})^k{}_c$$

RG flow seen along the eigendirections around the fixed point up to second order

$$S = S_* + \sum_a \mu^{\theta_a} \lambda^a \int \mathrm{d}^d x \, \mathcal{O}_a(x) + O(\lambda^2) \, .$$

$$\beta^a = -(d - \Delta_a)\lambda^a + \sum_{b,c} \tilde{C}^a{}_{bc} \, \lambda^b \lambda^c + O(\lambda^3) \, .$$

Take home message

one can extract not only the scaling dimensions, but also, reversing an argument from Cardy for a CFT, some OPE coefficients (structure constants) at order O(ε)

$$\langle \mathcal{O}_a(x) \, \mathcal{O}_b(y) \cdots \rangle = \sum_c \frac{1}{|x-y|^{\Delta_a + \Delta_b - \Delta_c}} \, C^c{}_{ab} \, \langle \mathcal{O}_c(x) \cdots \rangle$$

Linear term coefficients transform homogeneously
Quadratic term coefficients transform inhomogeneously

Possible scheme dependence!

Perturbative interlude: Ising Universality Class

ε-expansion below d=4 for the LG critical model $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + g\phi^4$

Leading counterterms in perturbation theory at order g^2 , dim reg $\overline{\mathrm{MS}}$

$d = 4 - \epsilon$

$$\mathcal{L}_{c.t.} = \frac{1}{\epsilon}\frac{1}{2(4\pi)^2}(12g)^2\phi^4 - \frac{1}{\epsilon}\frac{1}{6(4\pi)^4}(4!g)^2(\partial\phi)^2$$

$12g\phi^2$  $12g\phi^2$ $4!\,g\phi$  $4!\,g\phi$

Rescaling the coupling: $g \to (4\pi)^2 g$

beta function: $\beta_g = -\epsilon g + 72 g^2$

Two fixed points:

$g_* = 0$	$g_* = \frac{\epsilon}{72}$
UV gaussian	IR Wilson-Fisher

<u>Anomalous dimension</u>: $\eta = 2\tilde{\gamma}_1 = 96 g^2$ $\eta = \frac{\epsilon}{54}$

η is a universal quantity, independent from any coupling reparameterization!

Functional perturbative RG example: Ising UC

How to study deformations around the Wilson-Fisher fixed point? $d = 4 - \epsilon$

Couplings: $$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + g_1\phi + g_2\phi^2 + g_3\phi^3 + g_4\phi^4$$

Dimensionful beta functions
(global rescaling as before)

$$\beta_1 = 12\, g_2 g_3 - 108\, g_3^3 - 288\, g_2 g_3 g_4 + 48\, g_1 g_4^2$$
$$\beta_2 = 24\, g_4 g_2 + 18\, g_3^2 - 1080\, g_3^2 g_4 - 480\, g_2 g_4^2$$
$$\beta_3 = 72\, g_4 g_3 - 3312\, g_3 g_4^2$$
$$\beta_4 = 72\, g_4^2 - 3264\, g_4^3$$

At functional level:

$$\mathcal{L} = \frac{1}{2} Z(\phi)(\partial\phi)^2 + V(\phi)$$

1 loop — 2 loop

$$\beta_V = \frac{1}{2}\eta\phi V^{(1)} + a\frac{(V^{(2)})^2}{(4\pi)^2} + b\frac{V^{(2)}(V^{(3)})^2}{(4\pi)^4} + \cdots$$

$$\beta_Z = \eta Z + \frac{1}{2}\eta\phi Z^{(1)} + c\frac{(V^{(4)})^2}{(4\pi)^4} + \cdots$$

2 loop

$a = \frac{1}{2}$

$b = -\frac{1}{2}$

$c = -\frac{1}{6}$

Take home message

→

FPRG for multicritical models

Landau-Ginzburg lagrangian

$$S[\phi] = \int d^dx \left\{ \frac{1}{2}(\partial\phi)^2 + \mu^{\left(\frac{m}{2}-1\right)\epsilon} \frac{g}{m!}\phi^m \right\}$$

$$d = d_m - \epsilon$$

Upper critical dimension

$$d_m = \frac{2m}{m-2}$$

One marginal interaction at d_m .

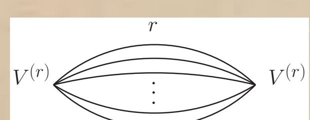
Study of deformations: we limit to a truncation $\mathcal{L} = \frac{1}{2} Z(\phi)(\partial\phi)^2 + V(\phi)$

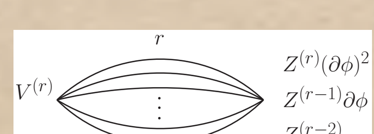
RG, even O'Dwyer, Osborn Ann. Phys. 323 (2008) 1859 Codello, Safari, G.P.V., Zanusso EPJ C78 (2018) 1

RG, odd Codello, Safari, G.P.V., Zanusso Phys. Rev. D98 (2017) 081701

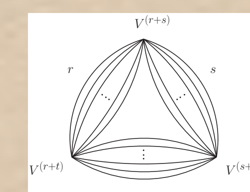
Multi-loop diagrams at functional level: $m = 2n$

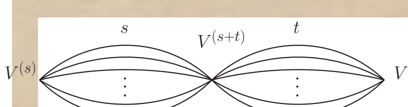
LO:

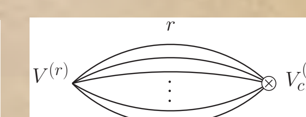




NLO:







$$\beta_v = -d\, v(\varphi) + \frac{d-2+\eta}{2}\varphi\, v'(\varphi) + \frac{n-1}{n!}\frac{c^{n-1}}{4} v^{(n)}(\varphi)^2$$
$$-\frac{n-1}{48} c^{2n-2}\Gamma(\delta_n) \sum_{\substack{r+s+t=2n \\ r,s,t \neq n}} \frac{K^n_{rst}}{r!s!t!} v^{(r+s)}(\varphi)\, v^{(s+t)}(\varphi)\, v^{(t+r)}(\varphi)$$
$$-\frac{(n-1)^2}{16\, n!} c^{2n-2} \sum_{s+t=n} \frac{n-1+L^n_{st}}{s!t!} v^{(n)}(\varphi)\, v^{(n+s)}(\varphi)\, v^{(n+t)}(\varphi)$$

$$\beta_z = \eta\, z(\varphi) + \frac{d-2+\eta}{2}\varphi\, z'(\varphi) - \frac{(n-1)^2}{(2n)!}\frac{c^{2n-2}}{4} v^{(2n)}(\varphi)^2$$
$$+ \frac{n-1}{n!}\frac{c^{n-1}}{2}\left[z^{(n)}(\varphi)\, v^{(n)}(\varphi) + z^{(n-1)}(\varphi)\, v^{(n+1)}(\varphi) \right]$$

Rescaling functions and fields to dimensionless quantities $v(\varphi), z(\varphi)$

FPRG for multicritical models

General pattern of mixing of the operators present in the truncation

$$
\begin{array}{lllllll}
V: & 1\ \phi\ \cdots\ \phi^{2n-1} & \phi^{2n} & \cdots & \phi^{4n-3} & \phi^{4n-2} & \cdots \\
Z: & & (\partial\phi)^2 & \cdots & \phi^{2n-3}(\partial\phi)^2 & \phi^{2n-2}(\partial\phi)^2 & \cdots \\
W_1: & & & & & \phi\Box^2\phi & \cdots \\
W_2: & & & & & (\partial_\mu\partial_\nu\phi)^2 & \cdots \\
W_3: & & & & & (\Box\phi)^2 & \cdots
\end{array}
$$

$$
\begin{pmatrix} M^{(0)} & & & \\ & M^{(2)} & & \\ & & M^{(4)} & \\ & & & \ddots \end{pmatrix}
$$

Anomalous dimensions of composite operators

$$\tilde{\gamma}_i = \frac{2(n-1)n!}{(2n)!}\frac{i!}{(i-n)!}\epsilon \qquad \tilde{\omega}_i = \frac{2(n-1)n!}{(2n)!}\frac{(i+1)!}{(i-n+1)!}\epsilon$$

At leading order the stability matrix M is triangular.

$$
\begin{aligned}
\tilde{\gamma}_i = {}& i\frac{\eta}{2} + \frac{(n-1)i!}{(i-n)!}\frac{2\,n!}{(2n)!}\left[\epsilon - \frac{n}{n-1}\eta\right] + 2n\,\eta\,\delta_i^{2n} \\
&+ \frac{(n-1)i!n!^6}{(2n)!^2}\Gamma(\delta_n)\sum_{\substack{r+s+t=2n \\ r,s,t\neq n}} \frac{K^n_{rst}}{(r!s!t!)^2}\left[\frac{2n!}{3(i-n)!} - \frac{r!}{(i-2n+r)!}\right]\epsilon^2 \\
&+ \frac{(n-1)^2 i!n!^5}{(2n)!^2}\sum_{s+t=n}\frac{n-1+L^n_{st}}{(s!t!)^2}\left[\frac{1}{(i-n)!} - \frac{2s!}{n!(i-2n+s)!}\right]\epsilon^2.
\end{aligned}
$$

OPE coefficients are read off the quadratic expansion of the beta functions

Other studies

- Multicritical higher derivative theories: there can be many marginal operators at criticality, results still to be understood in CFT.

Safari, G.P.V., Phys. Rev. D98 (2017) 081701, EPJ C78 (2018) 251

- Shift symmetric theories

Safari, G.P.V. in preparation

Multifield theories

- Potts models (cubic)

Osborn, Stergiou arXiv:1707.06165

Potts models (quintic)

Codello, Safari, G.P.V., Zanusso in preparation

Non trivial in d=3

Perturbative ε-expansion useful guide towards non perturbative regimes.

Non perturbative functional RG flows

Perturbation theory is very powerful to derive some qualitative informations even for infinite set of universal data, but for strongly interacting theories non perturbative tools are needed.

- Wilsonian flows:
  require the partition function to be independent from a UV cutoff.
  In general one can have

$$Z = \int [d\varphi] e^{-S_\Lambda[\varphi]}$$

$$\Lambda \frac{d}{d\Lambda} e^{-S_\Lambda[\varphi]} = \int dx \frac{\delta}{\delta\varphi(x)} \left(\psi_x^\Lambda[\varphi] e^{-S_\Lambda[\varphi]} \right)$$

$$\Lambda \frac{d}{d\Lambda} S_\Lambda[\varphi] = \int dx \left(\frac{\delta S_\Lambda[\varphi]}{\delta\varphi(x)} \psi_x^\Lambda[\varphi] - \frac{\delta \psi_x^\Lambda[\varphi]}{\delta\varphi(x)} \right)$$

In general the flow induced by coarse-graining corresponds to a non trivial action-dependent field redefinition

$$\varphi'(x) = \varphi(x) - \tfrac{\delta\Lambda}{\Lambda} \psi_x^\Lambda[\varphi]$$

Wilson-Polchinski RG flows

$$Z_{\Lambda_0}[J] = \int [d\varphi]\ e^{-\frac{1}{2}\varphi\cdot\Delta^{-1}\cdot\varphi - S^I_{\Lambda_0}[\varphi] + J\cdot\varphi}$$

Split in low (L) and high (H) energy modes

$$\varphi = \varphi_L + \varphi_H$$

$$\Delta = \Delta_L + \Delta_H$$

φ_L has support roughly for $|p| < \Lambda$

Integrating the high energy modes one defines the interacting action S^I_Λ from

$$e^{W_\Lambda[\varphi_L, J]} = Z_\Lambda[\varphi_L, J] = e^{-\frac{1}{2}J\cdot\Delta_H\cdot J + J\cdot\varphi_L - S^I_\Lambda[\Delta_H\cdot J + \varphi_L]}$$

It is flowing according to the Polchinski equation

$$\Lambda\frac{d}{d\Lambda}S^I_\Lambda[\varphi] = \frac{1}{2}\int dxdy\left(-\Lambda\frac{d}{d\Lambda}\Delta_H\right)_{xy}\left[\frac{\delta S^I[\varphi]}{\delta\varphi(y)}\frac{\delta S^I[\varphi]}{\delta\varphi(x)} - \frac{\delta^2 S^I[\varphi]}{\delta\varphi(y)\delta\varphi(x)}\right] + \text{const}$$

The partition function is independent from the UV cutoff

1PI effective average action RG flow

$$e^{-W_k[J]} = Z_k[J] = e^{-\Delta S_k[\frac{\delta}{\delta J}]} Z_k[J] = \int [d\varphi]\; e^{-S[\varphi] - \Delta S_k[\varphi] + J\cdot\varphi}$$

Infrared regulator: $\Delta S_k[\varphi] = \frac{1}{2}\varphi \cdot R_k \cdot \varphi$

$R_k(p^2) > 0$ for $p^2 \ll k^2$

$R_k(p^2) \to 0$ for $p^2 \gg k^2$

$R_k(p^2) \to \infty$ for $k \to \Lambda\; (\to \infty)$

Legendre transform

$$\Gamma_k[\phi] = \underset{J}{\mathrm{extr}}\,(J\cdot\phi - W_k[J]) - \Delta S_k[\phi]$$

$$e^{-\Gamma_k[\phi]} = \int [d\varphi]\; e^{-S[\varphi] + \frac{\delta \Gamma_k}{\delta \phi}\cdot(\varphi - \phi) - \Delta S_k[\varphi - \phi]}$$

Wetterich/Morris equation

$$\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_t R_k\right]$$

$t = \ln k/k_0$

Legendre type relation between Wilsonian action and effective average action

$$\Gamma_\Lambda[\varphi^c] + \frac{1}{2}(\varphi^c - \Phi)\cdot\Delta_H\cdot(\varphi^c - \Phi) = S^I_\Lambda[\Phi]$$

Multicritical Yukawa theories

Consider a QFT of one real scalar field and Dirac fermions (N_f, d_γ) Symmetries: $U(N_f) \quad Z_2$

$X_f = N_f \, d_\gamma$

Physics also for interacting fermion systems, for SUSY models/emergent susy, quark-mesonic interactions,…

We study the model at criticality with the flow of the effective average action in a local potential approximation (LPA) + (eventually) anomalous dimensions

Truncation: $$\Gamma_k[\phi, \psi, \bar{\psi}] = \int d^d x \left(\frac{1}{2} Z_{\phi,k} \, \partial^\mu \phi \partial_\mu \phi + V_k(\phi) + Z_{\psi,k} \bar{\psi} \gamma^\mu i \partial_\mu \psi + i \, H_k(\phi) \, \bar{\psi} \, \psi \right)$$

Two functions

G.P.V., Zambelli Phys. Rev. D91 (2015) 125003

Rescaled dimensionless quantities: $$v_k(\phi) = k^{-d} V_k \left(\frac{Z_\phi^{1/2} \phi}{k^{(d-2)/2}} \right) , \qquad h_k(\phi) = \frac{k^{-1}}{Z_\psi} H_k \left(\frac{Z_\phi^{1/2} \phi}{k^{(d-2)/2}} \right)$$

Flow equation for linear optimised regulators

$$\dot{v} = -dv + \frac{d-2+\eta_\phi}{2} \phi \, v' + C_d \left(\frac{1 - \frac{\eta_\phi}{d+2}}{1+v''} - X_f \frac{1 - \frac{\eta_\psi}{d+1}}{1+h^2} \right)$$

$$\dot{h} = h \, (\eta_\psi - 1) + \frac{d-2+\eta_\phi}{2} \phi \, h'$$
$$+ C_d \left[2h \, (h')^2 \left(\frac{1 - \frac{\eta_\psi}{d+1}}{(1+h^2)^2 \, (1+v'')} + \frac{1 - \frac{\eta_\phi}{d+2}}{(1+h^2) \, (1+v'')^2} \right) - \frac{h'' \left(1 - \frac{\eta_\phi}{d+2} \right)}{(1+v'')^2} \right]$$

Symmetries: v even and h odd. Parameters of the problem: (d, X_f)

Numerical analysis

Multicritical structure dictated by the <u>marginal interactions</u>, analysis with canonical dimensions

$$\phi^{2n} \quad : \qquad d_c^v(n \geq 2) = \frac{2n}{n-1} = 4, 3, \frac{8}{3}, \frac{5}{2}, \frac{12}{5} \cdots$$

$$\phi^{2n+1}\bar{\psi}\psi \quad : \qquad d_c^h(n \geq 0) = \frac{4(n+1)}{2n+1} = 4, \frac{8}{3}, \frac{12}{5} \cdots$$

Strategy:

- Numerical evolution from the origin
- Numerical evolution from the asymptotic region
- Polynomial truncations

Boundary conditions:

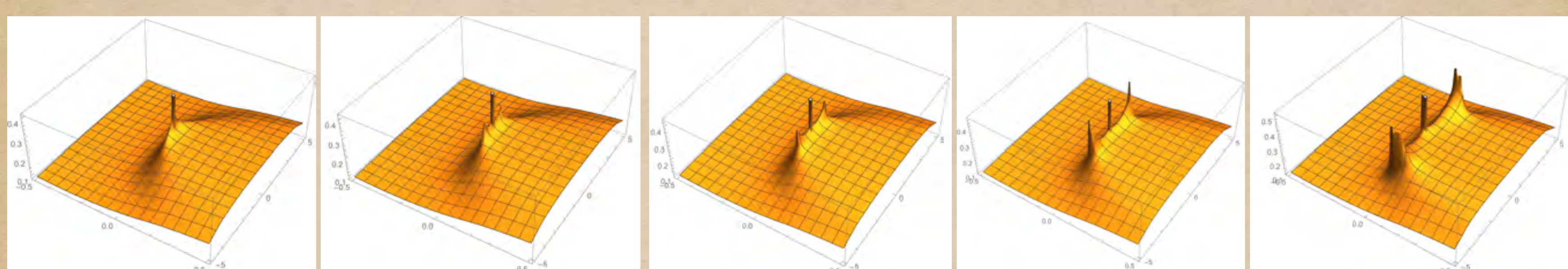
$v'(0) = 0 \qquad h(0) = 0$

$v''(0) = \sigma \qquad h'(0) = h_1$

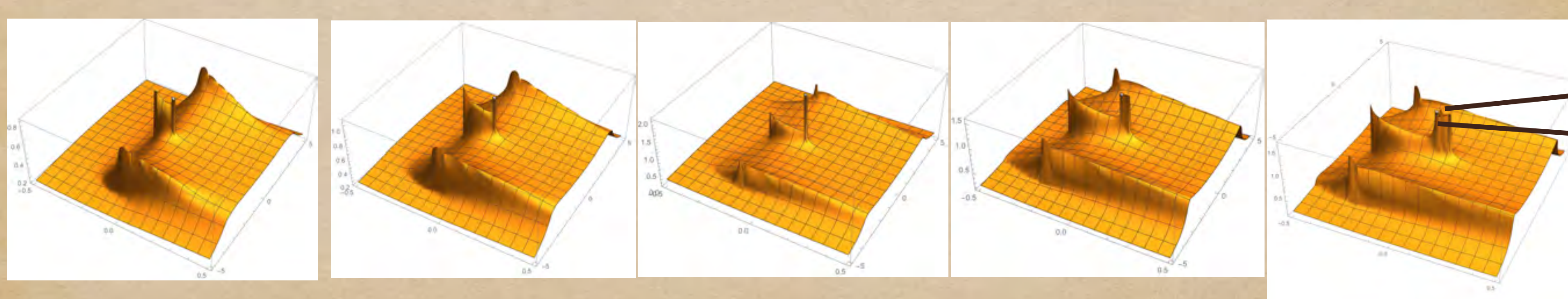
From the origin

$\frac{5}{2} < d < 4$

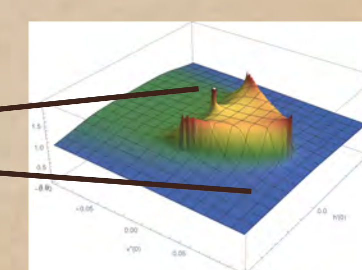
$X_f = 1$



d: 4 3.98 3.9 3.75 3.5



d: 3.25 3 2.9 8/3 2.57



These simple non linear ODEs already know pretty much!

Numerical analysis from the asymptotic region

At large field values one can construct the asymptotic expansion of the solution as a function of free parameters and then evolve numerically towards the origin imposing the known boundary conditions

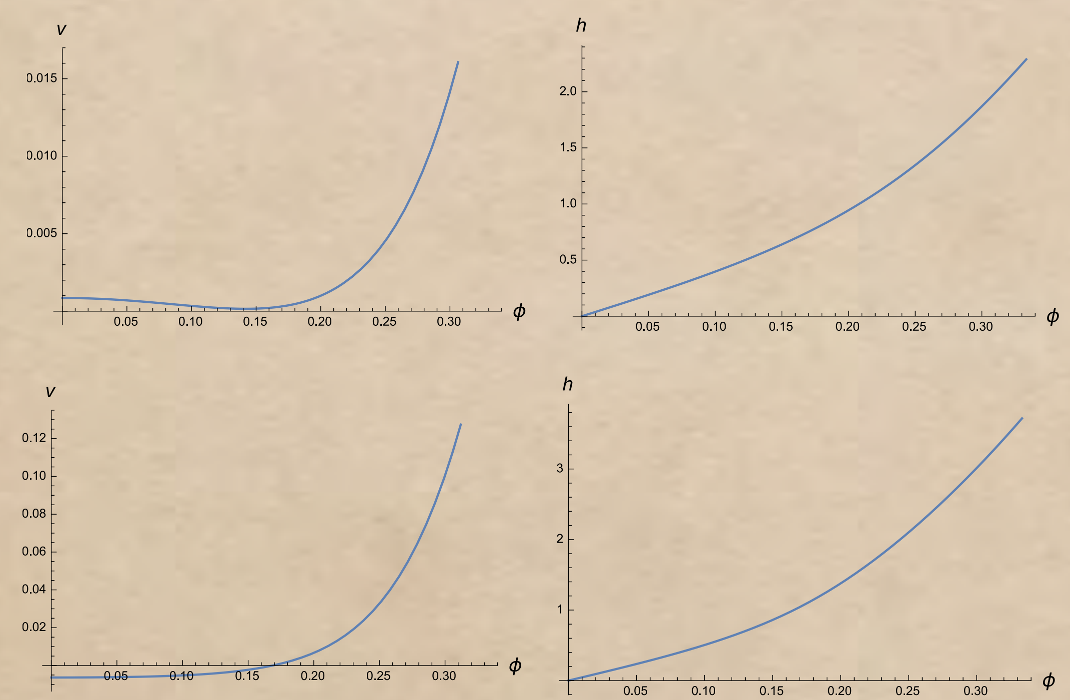
$$v'(0) = 0 \qquad h(0) = 0$$

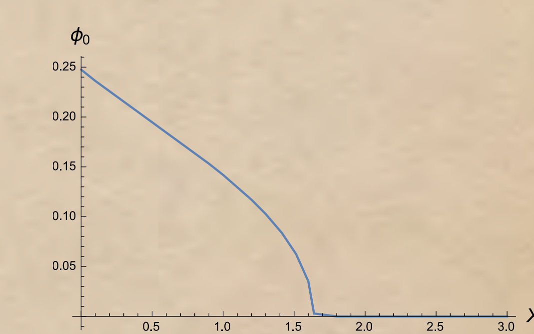
Some properties of the fully non trivial LPA scaling solutions in d=3:
if $X_f < 1.64$ the scalar is in the broken phase.

$v(\phi)$ $h(\phi)$

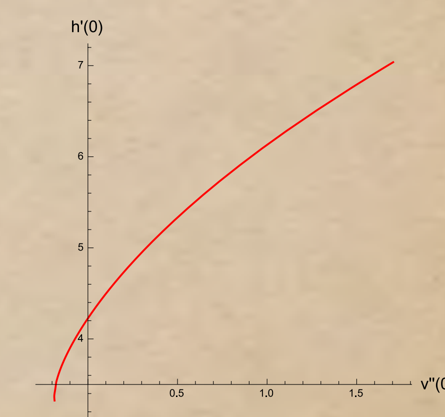
$X_f = 1$

$X_f = 2$





Vacuum



Locus of the solutions in the plane (σ, h_1) as function of X_f

Polynomial truncations

$$\rho = \phi^2/2 \qquad y(\rho) = h^2(\phi)$$

$$v(\rho) = \sum_{n=0}^{N_v} \frac{\lambda_n}{n!} \rho^n$$

$$v(\rho) = \lambda_0 + \sum_{n \geq 2}^{N_v} \frac{\lambda_n}{n!} (\rho - \kappa)^n$$

$$h(\phi) = \phi \sum_{n=0}^{N_h - 1} \frac{h_n}{n!} \rho^n$$

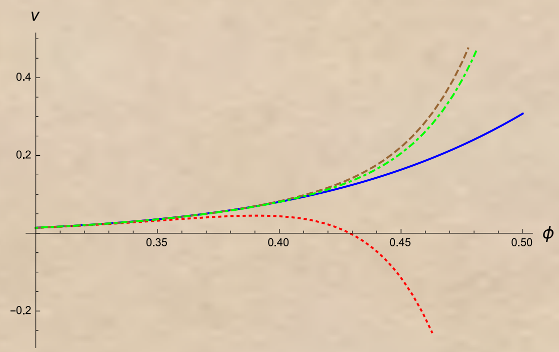
$$h(\phi) = \phi \sum_{n=0}^{N_h - 1} \frac{h_n}{n!} (\rho - \kappa)^n$$

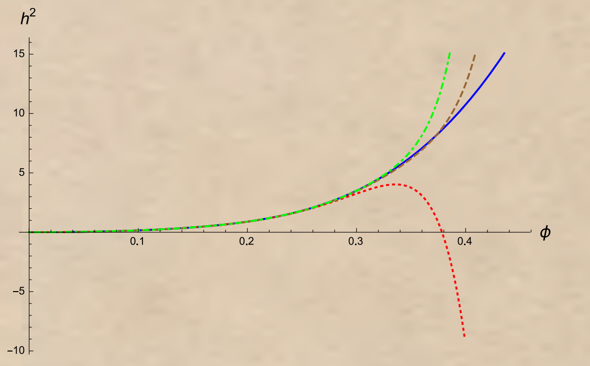
$$y(\rho) = \sum_{n=1}^{N_h} \frac{y_n}{n!} \left[(\rho - \kappa)^n - (-\kappa)^n \right]$$

Expansions: around the origin or non trivial vacuum (I,II) vs numerical ODE sol.

$N_v = 9$ $\quad X_f = 1$

$N_h = 8$





X_f	0.3	0.6	0.9	1.2	1.5	1.62
κ	$2.377\ 10^{-2}$	$1.793\ 10^{-2}$	$1.253\ 10^{-2}$	$7.315\ 10^{-3}$	$2.169\ 10^{-3}$	$1.125\ 10^{-4}$
λ_2	5.719	6.028	6.045	5.849	5.530	5.384
λ_3	55.00	61.19	61.55	57.37	50.79	47.90
y_1	17.51	15.62	13.67	11.85	10.26	9.690
y_2	214.7	192.0	162.1	131.55	104.5	95.07
θ_1	1.537	1.490	1.453	1.427	1.411	1.407
θ_2	−0.8152	−0.7882	−0.7755	−0.7751	−0.7831	−0.7877
θ_3	−0.9833	−1.066	−1.088	−1.062	−1.003	−0.9727
η_ψ	0.1510	0.1529	0.1537	0.1531	0.1514	0.1505
η_ϕ	0.1366	0.1687	0.2073	0.2499	0.2936	0.3108

X_f	1.62	2	3	4	6	8
λ_1	$-7.366\ 10^{-4}$	$4.137\ 10^{-2}$	0.1443	0.2316	0.3602	0.4448
λ_2	5.374	5.471	5.604	5.562	5.185	4.701
λ_3	47.81	43.63	32.95	23.64	11.05	4.560
y_1	9.667	9.304	8.296	7.338	5.804	4.733
y_2	94.77	83.91	59.23	41.28	20.95	11.67
θ_1	1.277	1.229	1.134	1.077	1.024	1.004
θ_2	−0.7775	−0.7742	−0.7794	−0.7962	−0.8345	−0.8649
θ_3	−0.8935	−0.9578	−1.101	−1.196	−1.287	−1.311
η_ψ	0.1508	0.1314	$9.347\ 10^{-2}$	$6.939\ 10^{-2}$	$4.341\ 10^{-2}$	$3.073\ 10^{-2}$
η_ϕ	0.3106	0.3721	0.5057	0.6024	0.7223	0.7894

Strong improvement comparing to results obtained with a smaller truncation with fixed $h(\phi) = h_1 \phi$

Moving to $Z_\phi(\phi)$ and $Z_\psi(\phi)$ not so useful, Knorr Phys. Rev. B94 (2016) 245102

probably needed 4 derivative expansion or momentum dependent vertex expansion.

Effective average Hamiltonian action

One can study the quantum/statistical field theory in phase space. G.P.V., Zambelli Phys. Rev. D86 (2012) 085041

$$S[p,q] = \int dt \left[p(t)\partial_t q(t) - H\left(p(t), q(t)\right) \right] \qquad e^{\frac{i}{\hbar}W[I,J]} = \int [dpdq]\, \mu[p,q] e^{\frac{i}{\hbar}\{S[p,q]+I\cdot p+J\cdot q\}}$$

$$\Gamma^H\left[\bar{p},\bar{q}\right] = \underset{I,J}{\mathrm{ext}}\left(W[I,J] - I\cdot\bar{p} - J\cdot\bar{q}\right) \qquad e^{\frac{i}{\hbar}\Gamma^H[\bar{p},\bar{q}]} = \int [dpdq]\, \mu[p,q] e^{\frac{i}{\hbar}\left\{S[p,q]-(q-\bar{q})\cdot\frac{\delta\Gamma^H}{\delta\bar{q}}-(p-\bar{p})\cdot\frac{\delta\Gamma^H}{\delta\bar{p}}\right\}}$$

Perturbative techniques are easily extended.

Wilsonian renormalization group for the action written in terms of the Hamiltonian.
Define an effective Hamiltonian flow. The coarse-graining is in the full phase space.

$$e^{iW_k[I,J]} = \int [dpdq]\, \mu_k[p,q] e^{i\{S[p,q]+\Delta S_k[p,q]+I\cdot p+J\cdot q\}} \qquad \Delta S_k[p,q] = \frac{1}{2}(p,q)\cdot R_k\cdot(p,q)^T$$

$$e^{i\Gamma_k[\bar{p},\bar{q}]} = \int [dpdq]\, \mu_k[p,q] e^{i\left\{S[p,q]+\Delta S_k[p-\bar{p},q-\bar{q}]-(p-\bar{p})\frac{\delta\Gamma_k}{\delta\bar{p}}-(q-\bar{q})\frac{\delta\Gamma_k}{\delta\bar{q}}\right\}}$$

$$\boxed{i\dot{\Gamma}_k = \frac{\dot{\mu}_k}{\mu_k} + i\langle \dot{\Delta S}_k[p-\bar{p},q-\bar{q}]\rangle_k} \qquad \Gamma[\bar{p},\bar{q}] = \int dt\,\left(\bar{p}\partial_t\bar{q} - H_k[\bar{p},\bar{q}]\right)$$

Example of regulators in (q,p)

$$R_k(t,t') = \begin{pmatrix} 0 & r_k(-\partial_t{}^2)\partial_t\delta(t-t') \\ -r_k(-\partial_t{}^2)\partial_t\delta(t-t') & 0 \end{pmatrix} \qquad \mu_k = \left[\mathrm{Det}\frac{1}{2\pi}\begin{pmatrix} 0 & \left(1+r_k(-\partial_t{}^2)\right)\partial_t\delta(t-t') \\ -\left(1+r_k(-\partial_t{}^2)\right)\partial_t\delta(t-t') & 0 \end{pmatrix}\right]^{\frac{1}{2}}$$

$$R_k(t,t') = \begin{pmatrix} \mathcal{R}^p_k(-\partial_t{}^2)\delta(t-t') & 0 \\ 0 & \mathcal{R}^q_k(-\partial_t{}^2)\delta(t-t') \end{pmatrix} \qquad \mu_k = \left[\mathrm{Det}\frac{1}{2\pi}\begin{pmatrix} \mathcal{R}^p_k(-\partial_t{}^2)\delta(t-t') & \partial_t\delta(t-t') \\ -\partial_t\delta(t-t') & \mathcal{R}^q_k(-\partial_t{}^2)\delta(t-t') \end{pmatrix}\right]^{\frac{1}{2}}.$$

Flow equations

Local Hamiltonian approximation (constant $\bar{q}$, $\bar{p}$)

Off-diagonal IR regulator

$$\det H_k^{(2)} = \partial^2_{\bar{q}\bar{q}} H_k \, \partial^2_{\bar{p}\bar{p}} H_k - (\partial^2_{\bar{q}\bar{p}} H_k)^2$$

costant

$$\frac{dH_r}{dr} = -\frac{1}{2(1+r)^2} \left(\det H_r^{(2)}\right)^{\frac{1}{2}}$$

optimized

$$\dot{H}_k = -\frac{k}{\pi} \frac{\det H_k^{(2)}}{k^2 + \det H_k^{(2)}}$$

One can study the spectrum of the quantum mechanical models non quadratic in the momenta, which have a non reducible path integral.

Diagonal IR regulator

$$\partial_{\mathcal{R}} \dot{H}_{\mathcal{R}} = -\frac{1}{\pi} \arctan\left(\frac{\Lambda}{\mathcal{R}}\right) + \frac{2\mathcal{R} + \partial^2_{\bar{p}\bar{p}} H_{\mathcal{R}} + \partial^2_{\bar{q}\bar{q}} H_{\mathcal{R}}}{2\pi \mathcal{D}_{\mathcal{R}}} \arctan\left(\frac{\Lambda}{\mathcal{D}_{\mathcal{R}}}\right)$$

$$\mathcal{D}_{\mathcal{R}} = \sqrt{\mathcal{R}^2 + \mathcal{R}\left(\partial^2_{\bar{p}\bar{p}} H_{\mathcal{R}} + \partial^2_{\bar{q}\bar{q}} H_{\mathcal{R}}\right) + \det H_{\mathcal{R}}^{(2)}}$$

A quantum mechanical example

To integrate the flow from the UV to the IR we need to specify the bare Hamiltonian at the UV scale. This is in 1-1 correspondence with Hamiltonian operator, being its Weyl symbol (i.e. Weyl ordered).

$$\langle x|\hat{O}|y\rangle = \int dp\, \langle x|p\rangle\, O_W\left(p, \frac{x+y}{2}\right) \langle p|y\rangle \qquad O_W(p,q) = \int dx\, e^{ipx} \langle q - \frac{x}{2}|\hat{O}(\hat{p},\hat{q})|q + \frac{x}{2}\rangle$$

First example for $H_n(p,q) = \left(\frac{p^2+\omega^2 q^2}{2}\right)^n$

Weyl symbol $\quad H_{2W}(p,q) = \left(\frac{p^2+q^2}{2}\right)^2 - \frac{1}{4} \quad , \quad H_{3W}(p,q) = \left(\frac{p^2+q^2}{2}\right)^3 - \frac{5}{4}\left(\frac{p^2+q^2}{2}\right)$

From numerical evolution one gets the effective Hamiltonian.
Numerical error in the spectrum <0.1%

Bare Hamiltonian	E_0^{exact}	E_0	E_0^{+1}	$\Delta E_1^{\text{exact}}$	ΔE_1	ΔE_1^{+1}
H_{2W}	1/4	0.24936	0.24936	2	1.99871	1.99871
H_2	1/2	0.49989	0.49994	2	1.99867	1.99985
H_{3W}	1/8	0.12492	0.124886	13/4	3.24736	3.24905
H_3	3/4	0.749849	0.74856	9/2	4.4991	4.4939

Diagonal cutoff schemes seem to work better.

Another example:

The Hamiltonian $\quad H_n(p,q) = p^n + a\, q^n \quad$ has instead the same Weyl symbol
Similar agreement.

Regge limit of strong interactions
Pomeron-Odderon Reggeon Field Theory

The main physical motivation is the idea that QCD, in the high energy (Regge) limit and at large distances, can be described by an effective theory such as Reggeon Field Theory (RFT), with local fields and local interactions.

$s \to \infty$

$t \simeq 0$

- Possible transition from QCD to the RFT regime:
  - BFKL physics: fundamental gluon (and quarks) organise themselves in composite fields (of reggeized gluons) giving as leading color singlet objects interacting Pomeron and Odderon,
    BFKL Pomeron ($J > 1$), Odderon ($J \simeq 1$) and both $\alpha' \simeq 0$
  - This should be at the "UV" boundary of RFT, below which (at larger distances) they may be considered approximately local with $J \simeq 1$ and a non zero α' and described by Regge poles, as in old S-matrix analysis of strong interactions intrinsically non perturbative.
- The onset of such a transition should involve mainly perturbative physics.
- Here we investigate some features of RFT in 2 transverse dimensions

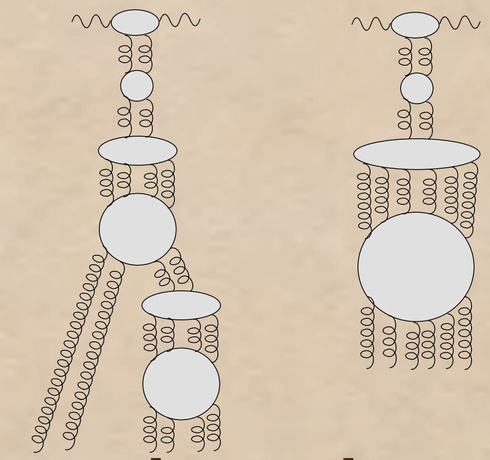
QCD in the Regge limit.

In early QCD times perturbative BFKL analysis found gluon reggeization, the Pomeron, as a composite state ψ of 2 reggeized gluons Lipatov et al. (1977)

and later the Odderon (C,P odd), as a composite state χ of 3 reggeized gluons, solution of the BKP equation in the lowest non trivial approximation. Bartels, Lipatov, G.P.V. (2000)

Simple exchanges of such objects are corrected by interactions in presence of more reggeized gluons in the t channel which are necessary to unitarize the theory.

Diagrams with reggeized gluons containing PPP and POO vertices: interactions are local in rapidity but non local in transverse space.



$$\frac{\partial N}{\partial \tau} = KN - V_{PPP} NN + V_{POO} OO$$
$$\frac{\partial O}{\partial \tau} = KO - V_{OPO}(NO + ON)$$

Approx. evolution in rapidity

Similar objects are found in other approaches to the Regge limit of QCD: CGC, Dipole/Wilson lines.

RFT might appear at high energies (large rapidities) and large transverse distances.

Odderon recently in the news because of TOTEM measurements at LHC!

Strong interactions and old Regge theory

About half a century ago V.N. Gribov introduced phenomenologically the RFT.

Starting point: Sommerfeld-Watson representation of the elastic scattering amplitudes.

$$\mathcal{T}_{AB}(s,t) = \int \frac{\mathrm{d}\omega}{2\mathrm{i}} \xi(\omega) s^{1+\omega} \mathcal{F}(\omega, t).$$

$$\xi(\omega) = \frac{\tau - \mathrm{e}^{-\mathrm{i}\pi\omega}}{\sin \pi\omega}$$

$$\tau = \pm 1$$

- Regge pole description in the complex $\omega = J - 1$ plane
- The leading pole: even signatured Pomeron with vacuum quantum numbers, trajectory $\alpha(t) \simeq \alpha_0 + \alpha' t$
- Unitarity in the crossed (t-channel): multi pomeron states, branch-point singularities (Regge cuts)
- Analysis of experimental inclusive cross sections in the triple Regge region showed that a triple Pomeron interaction should be introduced.
- In the '70 it was conjectured that another pole with odd quantum numbers (P,C, τ) could exist, the so called Odderon with $\alpha(0)$ close to 1.
- The Pomeron RFT was found to be in the same universality class as directed percolation. Cardy (1980),
  Non perturbative FRG analysis give good results!

Canet, et al. (2004(,

Bartels, Contreras, G.P.V. (2016),

RFT with Pomeron and Odderon fields

Symmetries

Interactions are constrained by signature: conservation
Reggeons have different signature factors,
multi reggeon cut has discontinuity with overall sign from $-\,\mathrm{i}\Pi_j(\mathrm{i}\xi_j)$

Pomeron: $\xi \simeq i$ (imaginary) , Odderon: $\xi \simeq -\frac{2}{\pi\omega}$ (real)

Couplings can be real or imaginary!

- n Pomeron t-channel states induced by interactions gets a factor $(-1)^{n-1}$
  Therefore the pomeron self energy is negative.
  The triple Pomeron coupling by convention is chosen imaginary.
  Quartic Pomeron couplings are real.

- Odderon has negative signature:
  transition $P \to OO$ is real valued; transition $O \to OP$ is imaginary
  Quartic interactions: most coupling remain real, but
  $O \to OOO$ and $P \to P + OO$ have imaginary coupling

Local effective action for RFT

$$\Gamma[\psi^\dagger,\psi,\chi^\dagger,\chi] = \int \mathrm{d}^D x \, \mathrm{d}\tau \left(Z_P(\frac{1}{2}\psi^\dagger \overleftrightarrow{\partial}_\tau \psi - \alpha'_P \psi^\dagger \nabla^2 \psi) + Z_O(\frac{1}{2}\chi^\dagger \overleftrightarrow{\partial}_\tau \chi - \alpha'_O \chi^\dagger \nabla^2 \chi) + V_k[\psi,\psi^\dagger,\chi,\chi^\dagger] \right)$$

- Allowed cubic interactions

Hamiltonian form

$$V_3 = -\mu_P \psi^\dagger \psi + i\lambda \psi^\dagger (\psi + \psi^\dagger)\psi - \\ -\mu_O \chi^\dagger \chi + i\lambda_2 \chi^\dagger (\psi + \psi^\dagger)\chi + \lambda_3 (\psi^\dagger \chi^2 + {\chi^\dagger}^2 \psi)$$

Bartels, Contreras, G.P.V. (2017),

- Allowed quartic interactions

$$V_4 = \lambda_{41}(\psi\psi^\dagger)^2 + \lambda_{42}\psi\psi^\dagger(\psi^2 + {\psi^\dagger}^2) + \lambda_{43}(\chi\chi^\dagger)^2 + i\lambda_{44}\chi\chi^\dagger(\chi^2 + {\chi^\dagger}^2) \\ + i\lambda_{45}\psi\psi^\dagger(\chi^2 + {\chi^\dagger}^2) + \lambda_{46}\psi\psi^\dagger\chi\chi^\dagger + \lambda_{47}\chi\chi^\dagger(\psi^2 + {\psi^\dagger}^2)$$

- …

- States with even and odd Odderon number do not mix.
- The couplings λ_3 and similarly λ_{44} and λ_{45} play a special role: they are responsible for the change of the Odderon number

We shall study the RG flow equation for a generic potential expanded as polynomial in the weak field approximation.

We shall consider a generic D dimensional transverse space but mainly work in D=2.

RTF: construction of the flow equations

General strategy used here for a polynomial truncation of the potential.

$$[\Gamma^{(2)}+\mathbb{R}]^{-1} = [\Gamma^{(2)}_{free} - V_{int}]^{-1}$$
$$= G(\omega,q) + G(\omega,q)V_{int}G(\omega,q) + G(\omega,q)V_{int}G(\omega,q)V_{int}G(\omega,q) + ...$$

$$G(\omega,q) = \begin{pmatrix} G_P(\omega,q) & 0 \\ 0 & G_O(\omega,q) \end{pmatrix}$$

$$G_P(\omega,q) = \begin{pmatrix} 0 & (Z_P(-i\omega+\alpha'_P q^2)+R_P-\mu_P)^{-1} \\ (Z_P(i\omega+\alpha'_P q^2)+R_P-\mu_P)^{-1} & 0 \end{pmatrix}$$

$$G_O(\omega,q) = \begin{pmatrix} 0 & (Z_O(-i\omega+\alpha'_O q^2)+R_O-\mu_O)^{-1} \\ (Z_O(i\omega+\alpha'_O q^2)+R_O-\mu_O)^{-1} & 0 \end{pmatrix}$$

$$V_{int} = -\begin{pmatrix} V^r_{\psi\psi} & V^r_{\psi\psi^\dagger} & V^r_{\psi\chi} & V^r_{\psi\chi^\dagger} \\ V^r_{\psi^\dagger\psi} & V^r_{\psi^\dagger\psi^\dagger} & V^r_{\psi^\dagger\chi} & V^r_{\psi^\dagger\chi^\dagger} \\ V^r_{\chi\psi} & V^r_{\chi\psi^\dagger} & V^r_{\chi\chi} & V^r_{\chi\chi^\dagger} \\ V^r_{\chi^\dagger\psi} & V^r_{\chi^\dagger\psi^\dagger} & V^r_{\chi^\dagger\chi} & V^r_{\chi^\dagger\chi^\dagger} \end{pmatrix}$$

IR regulator for the coarse-graining:

$$R_P(q^2) = Z_P\alpha'_P(k^2-q^2)\Theta(k^2-q^2),$$
$$R_O(q^2) = Z_O\alpha'_O(k^2-q^2)\Theta(k^2-q^2) = rZ_O\alpha'_P(k^2-q^2)\Theta(k^2-q^2) \qquad r = \frac{\alpha'_O}{\alpha'_P}$$

Anomalous dimensions: $\eta_P = -\frac{1}{Z_P}\partial_t Z_P, \quad \eta_O = -\frac{1}{Z_O}\partial_t Z_O \qquad \zeta_P = -\frac{1}{\alpha'_P}\partial_t\alpha'_P, \quad \zeta_O = -\frac{1}{\alpha'_O}\partial_t\alpha'_O$

Dimensionless quantities: $\tilde\psi = Z_P^{1/2}k^{-D/2}\psi, \quad \tilde\chi = Z_O^{1/2}k^{-D/2}\chi. \qquad \tilde V = \frac{V}{\alpha'_P k^{D+2}}$

For example:

$$\tilde\mu_P = \frac{\mu_P}{Z_P\alpha'_P k^2}, \quad \tilde\mu_O = \frac{\mu_O}{Z_O\alpha'_P k^2},$$
$$\tilde\lambda = \frac{\lambda}{Z_P^{3/2}\alpha'_P k^2}k^{D/2}, \quad \tilde\lambda_{2,3} = \frac{\lambda_{2,3}}{Z_O Z_P^{1/2}\alpha'_P k^2}k^{D/2}$$

Classical scaling:

$$(-(D+2)+\zeta_P)\,\tilde V + \left(\frac{D}{2}+\frac{\eta_P}{2}\right)\left(\tilde\psi\frac{\partial\tilde V}{\partial\tilde\psi} + \tilde\psi^\dagger\frac{\partial\tilde V}{\partial\tilde\psi^\dagger}\right) + \left(\frac{D}{2}+\frac{\eta_O}{2}\right)\left(\tilde\chi\frac{\partial\tilde V}{\partial\tilde\chi} + \tilde\chi^\dagger\frac{\partial\tilde V}{\partial\tilde\chi^\dagger}\right)$$

Cubic truncation: beta functions

Performing the traces, the beta functions for dimensionless quantities are:

$$\dot{\mu}_P = (-2+\eta_P+\zeta_P)\mu_P + 2A_P\frac{\lambda^2}{(1-\mu_P)^2} - 2A_O r\frac{\lambda_3^2}{(r-\mu_O)^2}$$

$$\dot{\mu}_O = (-2+\eta_O+\zeta_P)\mu_O + 2(A_P+A_O r)\frac{\lambda_2^2}{(1+r-\mu_P-\mu_O)^2}$$

$$\dot{\lambda} = (-2+D/2+\zeta_P+\frac{3}{2}\eta_P)\lambda + 8A_P\frac{\lambda^3}{(1-\mu_P)^3} - 4A_O r\frac{\lambda_2\lambda_3^2}{(r-\mu_O)^3}$$

$$\begin{aligned}\dot{\lambda}_2 = {} & (-2+D/2+\zeta_P+\frac{1}{2}\eta_P+\eta_O)\lambda_2 \\ & +\frac{2\lambda\lambda_2^2(6A_P+5A_O r)+4\lambda_3^2(A_P+A_O r)-4\lambda_2\lambda_3^2(A_P+2A_O r)}{(1+r-\mu_P-\mu_O)^3} \\ & +\frac{2A_P\lambda\lambda_2^2(r-\mu_O)^2}{(1-\mu_P)^2(1+r-\mu_P-\mu_O)^3} - \frac{4A_O r\lambda_2\lambda_3^2(1-\mu_P)^2}{(1-\mu_O)^2(1+r-\mu_P-\mu_O)^3} \\ & +\frac{2\lambda\lambda_2^2(3A_P+A_O r)(r-\mu_O)}{(1-\mu_P)(1+r-\mu_P-\mu_O)^3} - \frac{4\lambda_2\lambda_3^2(A_P+3A_O r)(1-\mu_P)}{(r-\mu_O)(1+r-\mu_P-\mu_O)^3}\end{aligned}$$

$$\begin{aligned}\dot{\lambda}_3 = {} & (-2+D/2+\zeta_P+\frac{1}{2}\eta_P+\eta_O)\lambda_3 \\ & +\frac{2\lambda_2^2\lambda_3(A_P+2A_O r)}{(r-\mu_O)(1+r-\mu_P-\mu_O)^2} + \frac{4\lambda\lambda_2\lambda_3(2A_P+A_O r)}{(1-\mu_P)(1+r-\mu_P-\mu_O)^2} \\ & +\frac{2\lambda_2^2\lambda_3 A_O r(1-\mu_P)}{(r-\mu_O)^2(1+r-\mu_P-\mu_O)^2} + \frac{4\lambda\lambda_2\lambda_3 A_P(r-\mu_O)}{(1-\mu_P)^2(1+r-\mu_P-\mu_O)^2}.\end{aligned}$$

$$\dot{r} = r\left(-\zeta_O+\zeta_P\right)$$

$$A_P = N_D A_D(\eta_P,\zeta_P), \quad A_O = N_D A_D(\eta_O,\zeta_O).$$

$$N_D = \frac{2}{\sqrt{4\pi}^D\,\Gamma(D/2)}$$

$$A_D(\eta_k,\zeta_k) = \frac{1}{D} - \frac{\eta_k+\zeta_k}{D(D+2)}$$

Similarly, one can find the anomalous dimensions (from the flow of 2-point functions):

$$\eta_P = -\frac{2A_P\lambda^2}{(1-\mu_P)^3} + \frac{2A_O r\lambda_3^2}{(r-\mu_O)^3}$$

$$\eta_O = -\frac{4(A_P+A_O\,r)\lambda_2^2}{(1+r-\mu_P-\mu_O)^3}$$

$$\eta_P+\zeta_P = -\frac{N_D\lambda^2}{D(1-\mu_P)^3} + \frac{N_D r^2\lambda_3^2}{D(r-\mu_O)^3}$$

$$\eta_O+\zeta_O = -\frac{4N_D\lambda_2^2}{D(1+r-\mu_P-\mu_O)^3}.$$

Perturbation theory: ϵ-expansion: $D = 4 - \epsilon$

Critical theory (fixed point): perturbative one loop results:

$$\mu_P = \frac{\epsilon}{12}, \quad \lambda^2 = \frac{8\pi^2}{3}\epsilon, \quad \eta_P = -\frac{\epsilon}{6}, \quad \zeta_P = \zeta_O = \frac{\epsilon}{12},$$

$$\mu_O = \frac{95+17\sqrt{33}}{2304}\epsilon, \quad \lambda_2^2 = \frac{23\sqrt{6}+11\sqrt{22}}{48}\epsilon, \quad \lambda_3 = 0, \quad \eta_O = -\frac{7+\sqrt{33}}{72}\epsilon, \quad r = \frac{3}{16}(\sqrt{33}-1)$$

Critical exponents: two relevant directions

$$\begin{aligned} \alpha_1 &= -2 + \frac{\epsilon}{4} \rightarrow \nu_P = \frac{1}{2} + \frac{\epsilon}{16} \\ \alpha_2 &= -2 + \frac{\epsilon}{12} \rightarrow \nu_O = \frac{1}{2} + \frac{\epsilon}{48}. \end{aligned}$$

The coupling of the changing Odderon number operator is zero!
The P ⟶ OO transition present in perturbative QCD is irrelevant and disappears.
Suppression of high mass diffractive scattering processes.

The Pomeron sector is not affected by the presence of the Odderon!

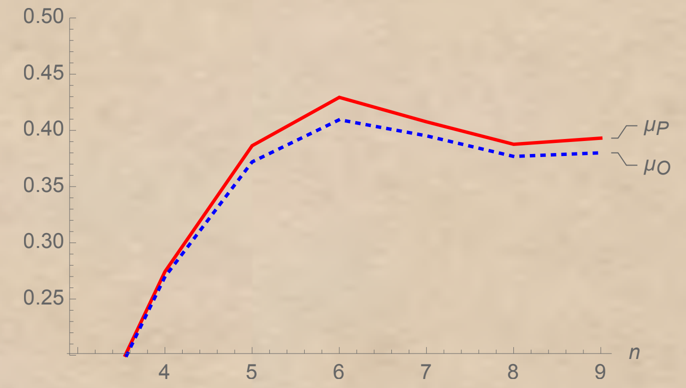
These qualitative features are maintained at non perturbative level!

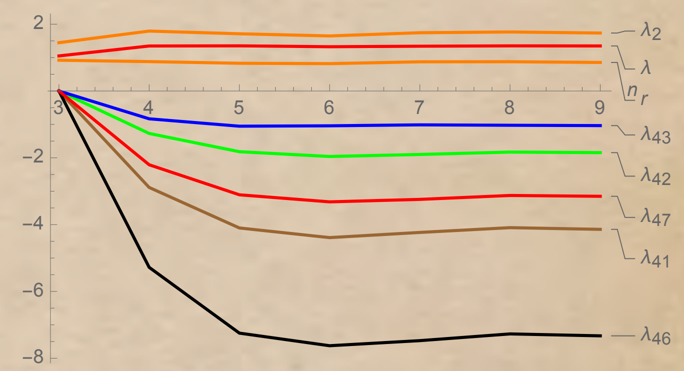
Non perturbative analysis in $D=2$

Explicit analysis at order 3,4,5 of the fixed points seems to show that the interactions changing the Odderon number are absent in the critical theory.

We perform the analysis of the fixed point up to order 9, neglecting (apart in r) the anomalous dimensions.

Couplings: fixed point values are stable at order 9.

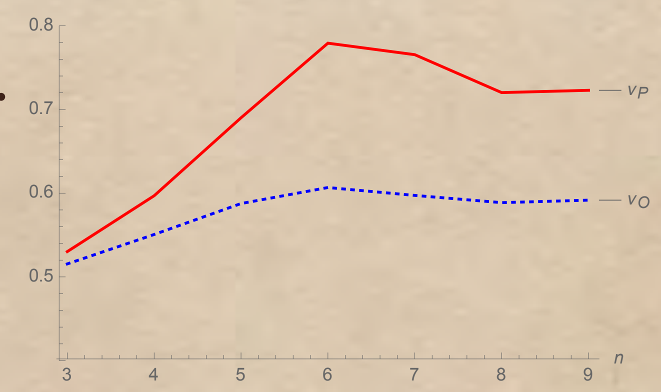


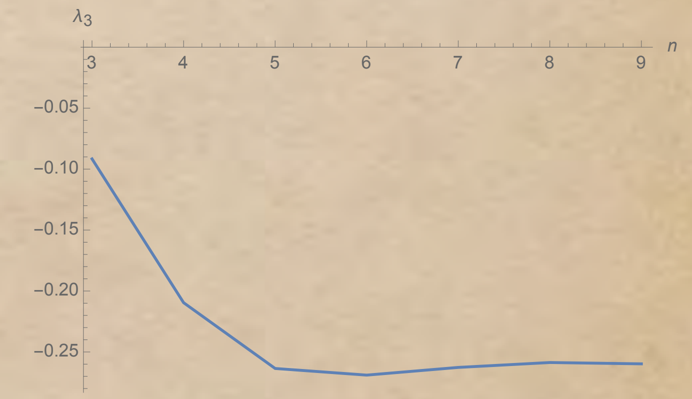


We find three relevant directions.
Critical exponents: $\nu = -1/\lambda$

$\nu_P \simeq 0.73,\ \nu_O \simeq 0.6$

$\lambda_3 \simeq -0.26$





Anomalous dimensions (cubic truncation estimate, close to ϵ-expansion result):

$\eta_P \simeq -0.33,\ \eta_O \simeq -0.35 \text{ and } \zeta_P = \zeta_O \simeq +0.17$

Conclusions and outlook

- Functional renormalization group is a powerful tool not yet fully exploited to study both critical and off-critical QFTs.
- It can be used both at perturbative and non perturbative (wilsonian) level
- In perturbation theory it is possible to directly compare or complement results with ones from CFT techniques (conformal universal data).
- At non perturbative level one has scheme dependent exact RG flow equations.
- Main problem: choice of truncations and approximations!
  Still new ideas are needed for a systematic control of the convergence.
- Gauge theories still harder to investigate at accurate level
- In many cases gives results at the level of montecarlo analysis for strongly interacting theories.
- At theoretical level tool to study the (geometry of) theory space of QFTs