Thermodynamic equilibrium with acceleration and the Unruh effect

F. B., arXiv:1712.08031 to appear in Phys. Rev. D

OUTLINE

- General global thermodynamic equilibria in flat spacetime
- Thermodynamic equilibrium with acceleration
- (Scalar) Quantum field theory in Rindler coordinates
- Thermal expectation values and vacuum subtraction

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Motivations

- Quantum field theory in nontrivial and local thermal equilibrium
- Description of fluids in local equilibrium with large accelerations (QGP in heavy ion collisions has initial acceleration $a \sim 10^{30}$ g)
- Stress-energy tensor in general relativity beyond ideal fluid approximation including quantum effects

Mean values

In a quantum statistical framework, the stress-energy tensor is defined as:

$$T^{\mu\nu}(x) = \mathrm{tr}(\widehat{\rho}\widehat{T}^{\mu\nu}(x))_{\mathrm{ren}}$$

The density operator of the familiar global thermodynamical equilibrium in flat spacetime (in covariant form):

$$\widehat{\rho} = (1/Z)\exp[-\beta\cdot\widehat{P} + \zeta\widehat{Q}]$$

$$\beta^\mu = \frac{1}{T}u^\mu$$

$$T = 1/\sqrt{\beta^2}$$

$$\zeta = \mu/T$$

$$T^{\mu\nu}(x) = (\rho+p)u^\mu u^\nu - pg^{\mu\nu}$$

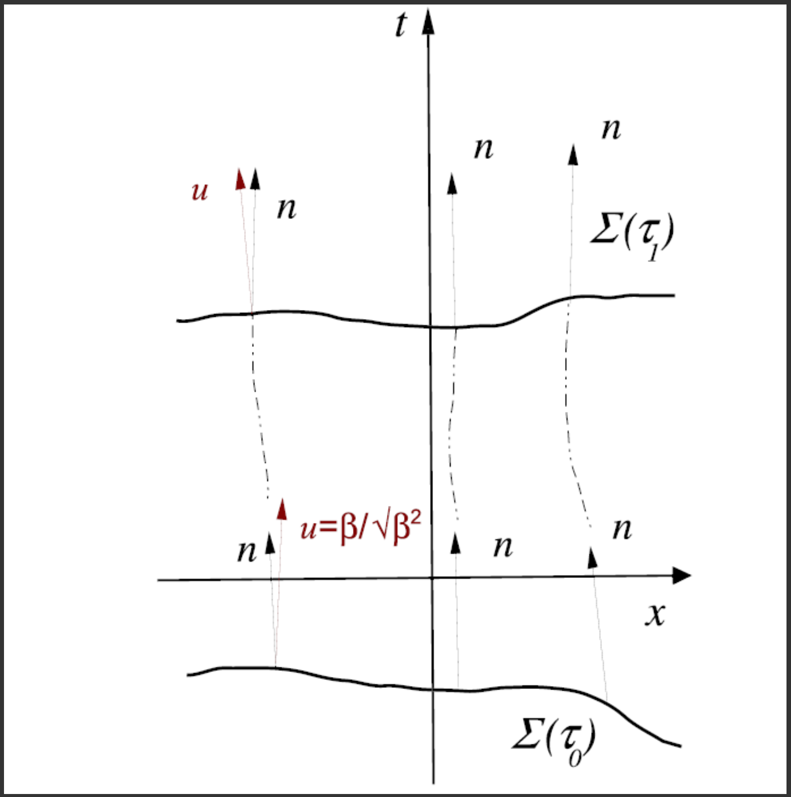
$\rho = \rho(T,\mu) = \rho(\beta^2,\zeta)$ energy density

General covariant (local) equilibrium

Zubarev 1979   Weert 1982

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_\Sigma \mathrm{d}\Sigma_\mu \, \left(\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

This operator is obtained by maximizing the entropy with the constraints of energy density and momentum density



F. B., L. Bucciantini, E. Grossi, L. Tinti,
Eur. Phys. J. C 75 (2015) 191
(β frame)

T. Hayata, Y. Hidaka, T. Noumi, M. Hongo,
Phys. Rev. D 92 (2015) 065008

General covariant *global* t.d. equilibrium in flat spacetime

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_\Sigma \mathrm{d}\Sigma_\mu \, \left(\widehat{T}^{\mu\nu}\beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

If the divergence of the integrand vanishes, that is:

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$$

$$\partial_\mu \zeta = 0$$

Σ can now be an arbitrary general timelike 3D hypersurface

For global equilibrium β (=*1/T u*) must be a Killing vector

Solution of the Killing equation in Minkowski spacetime:

$$\beta^\nu = b^\nu + \varpi^{\nu\mu} x_\mu$$

constant

$$\varpi_{\nu\mu} = -\frac{1}{2}(\partial_\nu \beta_\mu - \partial_\mu \beta_\nu)$$

Thermal vorticity

Adimensional in natural units

General global equilibrium -2

Plugging the solution into the general covariant expression of the density operator:

$$\widehat{\rho} = \frac{1}{Z} \exp \left[-b_\mu \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} + \zeta \widehat{Q} \right]$$

with

$$\widehat{J}^{\mu\nu} = \int_\Sigma \mathrm{d}\Sigma_\lambda \left(x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} \right)$$

Therefore, the most general thermodynamical equilibrium in Minkowski spacetime involves the 10 generators of its maximal symmetry group.

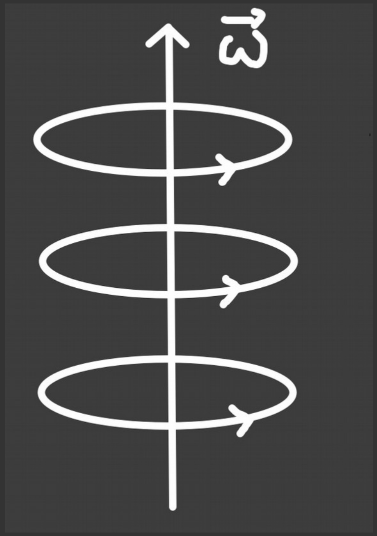
Special cases

Pure rotation (Landau *Statistical Physics*)

$$b_\mu = (1/T_0, 0, 0, 0) \qquad \varpi_{\mu\nu} = (\omega/T_0)(g_{1\mu}g_{2\nu} - g_{1\nu}g_{2\mu})$$

$$\beta^\mu = \frac{1}{T_0}(1, \boldsymbol{\omega} \times \mathbf{x})$$

$$\widehat{\rho} = (1/Z)\exp[-\widehat{H}/T_0 + \omega \widehat{J}_z/T_0]$$

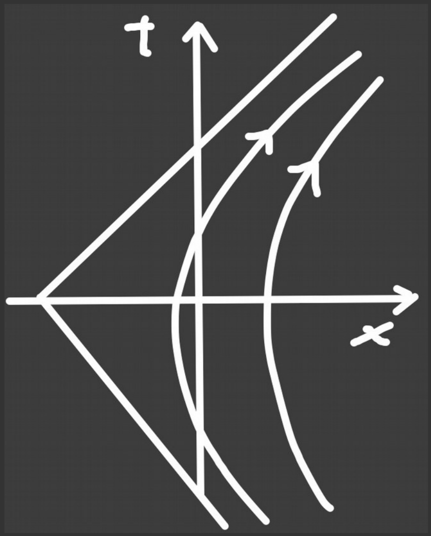


Pure acceleration (the subject of this talk)

$$b_\mu = (1/T_0, 0, 0, 0) \qquad \varpi_{\mu\nu} = (a/T_0)(g_{0\nu}g_{3\mu} - g_{3\nu}g_{0\mu})$$

$$\beta^\mu = \frac{1}{T_0}(1 + az, 0, 0, at)$$

$$\widehat{\rho} = (1/Z)\exp[-\widehat{H}/T_0 + a\widehat{K}_z/T_0]$$



What is it?

$$\widehat{\rho} = (1/Z) \exp[-\widehat{H}/T_0 + a\widehat{K}_z/T_0]$$

H and *K* are both constant (even if they do not commute)

$$i\frac{\mathrm{d}\widehat{K}_z}{\mathrm{d}t} = [\widehat{K}_z, \widehat{H}] + i\frac{\partial \widehat{K}_z}{\partial t} = -i\widehat{P}_z + i\widehat{P}_z = 0$$

At t=0

$$\widehat{H} - a\widehat{K}_z = \int \mathrm{d}^3\mathrm{x}\, (1 + az)\widehat{T}^{00}$$

Single non-relativistic particle (restoring c)

$$\widehat{H} - a\widehat{K}_z = (mc^2 + \widehat{p}^2/2m) \int \mathrm{d}^3\mathrm{x}\, (1 + az/c^2)\delta^3(\mathbf{x} - \widehat{\mathbf{x}}) \simeq mc^2 + \widehat{p}^2/2m + ma\widehat{z}$$

Hamiltonian of a particle in a constant and uniform gravitational field

Flow features

T_0, a constants

$$\beta^\mu = \frac{1}{T_0}(1 + az, 0, 0, at)$$

$$\beta^\mu = \frac{1}{T}u^\mu$$

Shift the origin: $z' = z - 1/a$

$$\beta^\mu = \frac{a}{T_0}(z', 0, 0, t)$$

Field lines are hyperbolae with constant $k = \sqrt{z'^2 - t^2}$

Velocity field

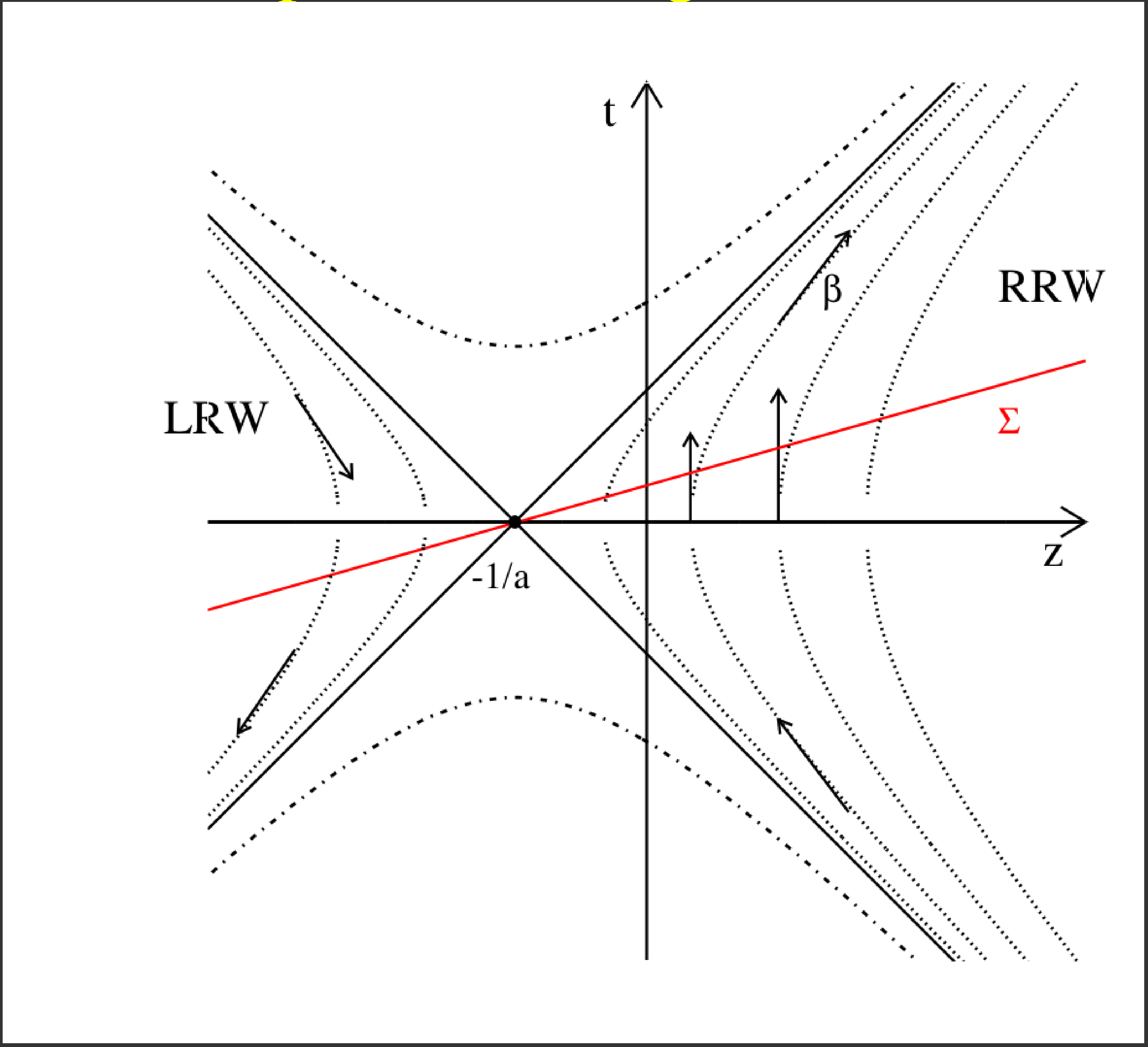
$$u^\mu = \frac{1}{k}(z', 0, 0, t)$$

Acceleration field

$$A^\mu = \frac{1}{k^2}(t, 0, 0, z')$$

Comoving acceleration A^2 constant along flow lines (relativistic uniformly accelerated motion)

Space time diagram



Thermal features

$$\beta^\mu = \frac{1}{T_0}(1 + az, 0, 0, at)$$

T_0, a constants

$$k = \sqrt{z'^2 - t^2}$$

Comoving temperature – constant along flow lines (implied by Killing equation)

$$T = \frac{1}{\sqrt{\beta^2}} = \frac{T_0}{ka}$$

Ratio between comoving acceleration and comoving temperature is constant

$$-\frac{A^2}{T^2} = \frac{a^2}{T_0^2}$$

Temperature measured by a thermometer at rest with the inertial observer = $1/\beta^0$

$$T_{\text{inertial}} = T_0 \frac{1}{az'} = T_0 \frac{1}{a\sqrt{k^2 + t^2}}$$

Rewriting the density operator
(no chemical potential for simplicity)

$$\widehat{\rho} = (1/Z) \exp\left[- \int_\Sigma \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \zeta \widehat{j}^\mu\right)\right]$$

$$\widehat{\rho} = (1/Z) \exp[-\widehat{H}/T_0 + a\widehat{K}_z/T_0]$$

$$\widehat{K}'_z = \widehat{K}_z - \frac{1}{a}\widehat{H}$$

$$\widehat{\rho} = \frac{1}{Z} \exp\left[a\widehat{K}'_z/T_0\right]$$

Because the Killing vector β or γ = β T_0 vanishes in z'=0

$$-a\widehat{K}'_z = \int \mathrm{d}\Sigma_\mu \widehat{T}^{\mu\nu}\gamma_\nu \equiv \widehat{\Pi} = \int_{z'>0} \mathrm{d}\Sigma_\mu \widehat{T}^{\mu\nu}\gamma_\nu + \int_{z'<0} \mathrm{d}\Sigma_\mu \widehat{T}^{\mu\nu}\gamma_\nu \equiv \widehat{\Pi}_R - \widehat{\Pi}_L$$

$$[\widehat{\Pi}_R, \widehat{\Pi}_L] = 0$$

Decoupling of RRW and LRW

Factorization of the density operator in the RRW and LRW

$$\widehat{\rho} = \frac{1}{Z} \exp[-\widehat{\Pi}_R/T_0] \exp[\widehat{\Pi}_L/T_0]$$

With operators acting on the Hilbert spaces of the field degrees of freedom in the RRW and LRW

$$\widehat{\Pi}_R = \widehat{\Pi}_R \otimes I \qquad \widehat{\Pi}_L = I \otimes \widehat{\Pi}_L$$

Partition function also factorizes:

$$Z = \mathrm{tr}(\exp[-(\widehat{\Pi}_R - \widehat{\Pi}_L)/T_0]) = \mathrm{tr}_R(\exp[-\widehat{\Pi}_R/T_0])\mathrm{tr}_L(\exp[\widehat{\Pi}_L/T_0])$$

The mean value of a local operator only involves the wedge *x* belongs to. If x ∈ RRW:

$$\langle \widehat{O}(x) \rangle \equiv \frac{1}{Z} \mathrm{tr}(\exp[-a\widehat{K}'_z]\widehat{O}(x)) = \frac{1}{Z_R} \mathrm{tr}_R(\widehat{O}(x) \exp[-\widehat{\Pi}_R/T_0])$$

Quantum field theory in Rindler coordinates

L.C.B. Crispino, A. Higuchi and G.E.A. Matsas, *The Unruh effect and its applications*, Rev. Mod. Phys. 80 (2008) 787

1 - Klein-Gordon inner product

$$(\phi_1, \phi_2) = i \int_\Sigma \mathrm{d}\Sigma_\mu \; (\phi_1^* \nabla^\mu \phi_2 - \phi_2 \nabla^\mu \phi_1^*)$$

Σ is the (arbitrary) spacelike quantization hypersurface

2 – Expand the field into (normalized) eigenfunctions of the KG equation, with positive and negative frequencies of the normal derivative $n_\mu \nabla^\mu$ at the hypersurface

$$\widehat{\psi}(x) = \sum_i u_i \widehat{a}^i + u_i^* \widehat{a}_i^\dagger$$

$$(u_i, u_j) = \delta_{ij} \implies (u_i^*, u_j^*) = -\delta_{ij} \qquad (u_i^*, u_j) = 0 \implies (u_i, u_j^*) = 0$$

3 – Enforce quantization

$$[\widehat{a}_i, \widehat{a}_j^\dagger] = \delta_{ij} \qquad [\widehat{a}_i, \widehat{a}_j] = [\widehat{a}_i^\dagger, \widehat{a}_j^\dagger] = 0$$

Note that:

$$\widehat{a}_j = (u_j, \widehat{\psi}) \qquad \widehat{a}_j^\dagger = -(u_j^*, \widehat{\psi})$$

Quantum field theory in Rindler coordinates (cont'd)

To calculate mean values with exp $(-\Pi/T_0)$, it is convenient to quantize in Rindler coordinates. This requires the introduction of two different coordinates set for the RRW and the LRW

$$t = \frac{\mathrm{e}^{a\xi}}{a}\sinh(a\tau) \qquad z' = \frac{\mathrm{e}^{a\xi}}{a}\cosh(a\tau)$$

RRW

$$t = -\frac{\mathrm{e}^{a\bar{\xi}}}{a}\sinh(a\bar{\tau}) \qquad z' = -\frac{\mathrm{e}^{a\bar{\xi}}}{a}\cosh(a\bar{\tau})$$

LRW

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \gamma^\mu$$

$$\widehat{\Pi} = T_0 \int \mathrm{d}\Sigma_\mu \widehat{T}^{\mu\nu}\beta_\nu = \int \mathrm{d}\Sigma_\mu \widehat{T}^{\mu\nu}\gamma_\nu$$

It can be shown that

$$[\widehat{\Pi}, \widehat{\psi}(x)] = -i\frac{\partial}{\partial\tau}\widehat{\psi}$$

→ Π is the generator of translations along the Killing field lines

The eigenfunctions are the same in both wedges, but the role of creation and destruction operators is interchanged because the positive time direction is opposite to τ in the LRW

$$\widehat{\psi}(\tau,\xi,\mathbf{x}_T)^{(R)} = \int_0^\infty \mathrm{d}\omega \int \mathrm{d}^2k_T \, \left(u_{\omega\mathbf{k}_T}\widehat{a}^{(R)}_{\omega\mathbf{k}_T} + u^*_{\omega\mathbf{k}_T}\widehat{a}^{\dagger(R)}_{\omega\mathbf{k}_T}\right)$$

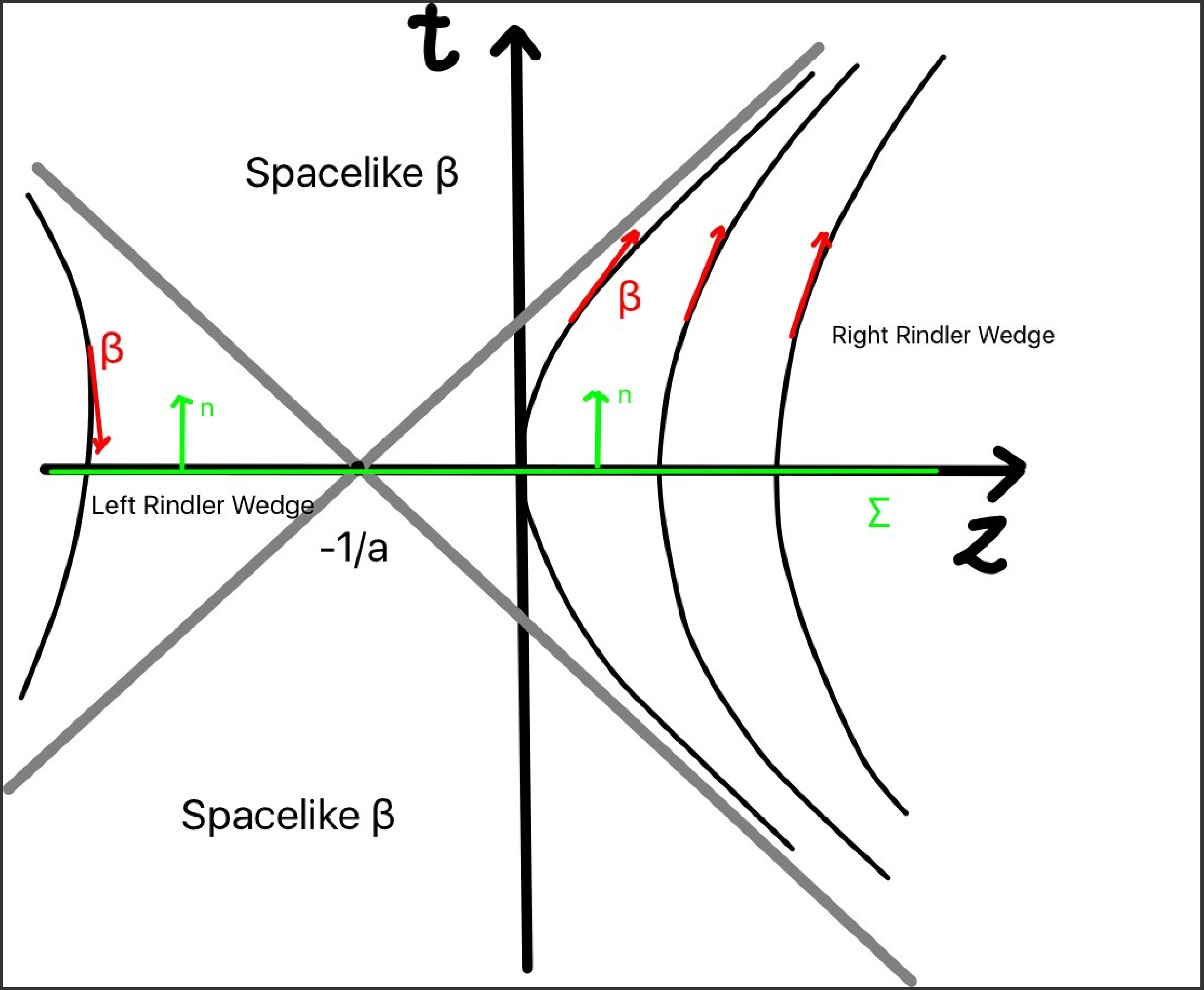
$$\widehat{\psi}(\tau,\xi,\mathbf{x}_T)^{(L)} = \int_0^\infty \mathrm{d}\omega \int \mathrm{d}^2k_T \, \left(u_{\omega\mathbf{k}_T}\widehat{a}^{\dagger(L)}_{\omega\mathbf{k}_T} + u^*_{\omega\mathbf{k}_T}\widehat{a}^{(L)}_{\omega\mathbf{k}_T}\right)$$

General eigenfunction

$$u(\tau,\xi,\mathbf{x}_T)_{\omega\mathbf{k}_T} = \sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}}\mathrm{K}_{i\omega/a}\left(\frac{m_T \mathrm{e}^{a\xi}}{a}\right)\mathrm{e}^{i\mathbf{k_T}\cdot\mathbf{x}_T}\mathrm{e}^{-i\omega\tau}$$

$$[\widehat{a}^{(R)}_{\omega\mathbf{k}_T},\widehat{a}^{\dagger(R)}_{\omega'\mathbf{k}'_T}] = \delta(\omega-\omega')\delta^2(\mathbf{k}_T-\mathbf{k}'_T)$$

Space time diagram



Thermal-acceleration field theory

Objective: calculate mean values of local operators with density operator

$$\widehat{\rho} = \frac{1}{Z}\exp[-\widehat{\Pi}/T_0]$$

1- Define inner operator product

$$(\widehat{\psi}_1, \widehat{\psi}_2) = i\int_\Sigma \mathrm{d}\Sigma_\mu \left(\widehat{\psi}_1^\dagger \nabla^\mu \widehat{\psi}_2 - \widehat{\psi}_1^\dagger \nabla^\mu \widehat{\psi}_2\right)$$

2- Show that

$$\widehat{\Pi} = \frac{i}{2}(\widehat{\psi}, \gamma\cdot\nabla\widehat{\psi}) = \frac{i}{2}(\widehat{\psi}, \frac{\partial}{\partial\tau}\widehat{\psi})$$

3 – Calculate Π:

$$\widehat{\Pi} = \frac{1}{2}\sum_i \omega_i \left(\widehat{a}_i^{\dagger(R)}\widehat{a}_i^{(R)} - \widehat{a}_i^{\dagger(L)}\widehat{a}_i^{(L)}\right)$$

Thermal expectation values of particle number operators

$$\widehat{\Pi} = \frac{1}{2}\sum_i \omega_i \left(\widehat{a}_i^{\dagger(R)}\widehat{a}_i^{(R)} - \widehat{a}_i^{\dagger(L)}\widehat{a}_i^{(L)}\right)$$

This form of the P operator makes it easy to determine the thermal expectation values (TEV) of quadratic combinations of Rindler creation and annihilation operators by using the Familiar method in thermal field theory:

In the RRW

$$\frac{1}{Z}\mathrm{tr}\left(\exp[-\widehat{\Pi}/T_0]\widehat{a}_i^{\dagger(R)}\widehat{a}_j^{(R)}\right) = \langle\widehat{a}_i^{\dagger(R)}\widehat{a}_j^{(R)}\rangle = \delta_{ij}\frac{1}{\mathrm{e}^{\omega/T_0}-1}$$

whereas in the LRW

$$\langle\widehat{a}_i^{\dagger(L)}\widehat{a}_j^{(L)}\rangle = \delta_{ij}\sum_{k=1}^{\infty}\mathrm{e}^{k\omega/T_0}$$

Renormalizing T.E.V.s

Any quadratic operator in the fields in the RRW will have a T.E.V.
where A and B are operations such as multiplication for a scalar or derivation.

$$\langle A\widehat{\psi}B\widehat{\psi}\rangle = \int_0^{+\infty} \mathrm{d}\omega \int \mathrm{d}^2\mathrm{k}_T \left[f_{\omega,\mathbf{k}_T} \frac{1}{\mathrm{e}^{\omega/T_0}-1} + f^*_{\omega,\mathbf{k}_T} \left(\frac{1}{\mathrm{e}^{\omega/T_0}-1} + 1 \right) \right]$$

This term gives rise to an infinite and must be renormalized

The usual renormalization in free-field theory is carried out by subtracting the *Minkowski vacuum contribution.* It seems reasonable to do the same here

$$\langle A\widehat{\psi}B\widehat{\psi}\rangle_{\mathrm{ren}} = \langle A\widehat{\psi}B\widehat{\psi}\rangle - \langle 0_{\mathrm{M}}|A\widehat{\psi}B\widehat{\psi}|0_{\mathrm{M}}\rangle$$

Renormalizing T.E.V.s and Unruh effect

These (Minkowski) V.E.V.s are the well known content of the Unruh effect:

$$\langle 0_{\rm M}|\widehat{a}_i^{\dagger(R)}\widehat{a}_j^{(R)}|0_{\rm M}\rangle = \langle 0_{\rm M}|\widehat{a}_i^{\dagger(L)}\widehat{a}_j^{(L)}|0_{\rm M}\rangle = \delta_{ij}\frac{1}{{\rm e}^{2\pi\omega_i/a}-1}$$

Therefore, the renormalization results in:

$$\langle A\widehat{\psi}B\widehat{\psi}\rangle_{\rm ren} = \int_0^{+\infty}{\rm d}\omega\int{\rm d}^2{\rm k}_T\;(f_{\omega,{\bf k}_T}(x)+f_{\omega,{\bf k}_T}(x)^*)\left(\frac{1}{{\rm e}^{\omega/T_0}-1}-\frac{1}{{\rm e}^{2\pi\omega/a}-1}\right)$$

Which means that the renormalized T.E.V. of any quadratic quantity vanishes when $T_0 = a/2\pi$

This conclusion extends to interacting field theories because (Bisognano Wichmann 1975)

$$\langle\widehat{O}(x)\rangle_{\rm ren} = {\rm tr}(\widehat{\rho}(T_0)\widehat{O}(x)) - \langle 0_{\rm M}|\widehat{O}(x)|0_{\rm M}\rangle = {\rm tr}(\widehat{\rho}(T_0)\widehat{O}(x)) - {\rm tr}((\widehat{\rho}(a/(2\pi))\widehat{O}(x))$$

Consequence

All quadratic operators (including the stress-energy tensor) have a vanishing mean value when $T_0 = 2\pi/a$ and not when $T_0 = 0$

Note that

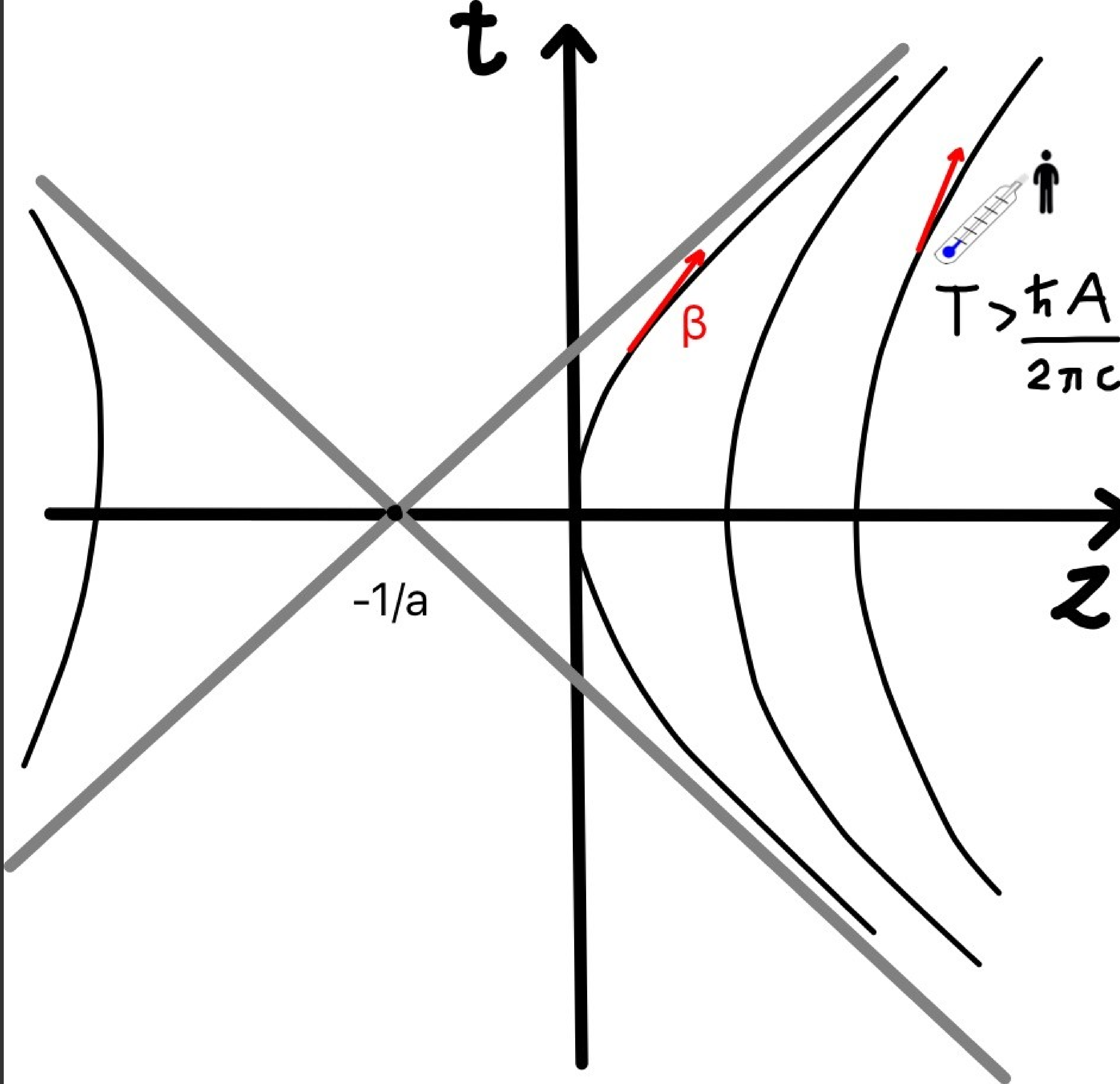
$$-\frac{A^2}{T^2} = \frac{a^2}{T_0^2}$$

so when $T_0 = 2\pi/a$

$$T = |A|/2\pi = T_U$$

Comoving thermometer sees comoving Unruh temperature

→ *An ideal thermometer moving along the accelerated world lines in the Minkowski vacuum state always marks a proper temperature equal to the magnitude of its proper acceleration divided by 2π. This must be an absolute lower bound.*



t
z
-1/a
β
T > ħA / 2πc

Lorentz invariance and an example

The T.E.V. of a Lorentz scalar can only depend on the proper *T* and *A*

$$F\left(T^2, \frac{A^2}{T^2}\right) = F\left(\frac{T_0^2}{k^2 a^2}, \frac{a^2}{T_0^2}\right)$$

$$F\left(\frac{T_0^2}{k^2 a^2}, \frac{a^2}{T_0^2}\right) - F\left(\frac{1}{(2\pi)^2 k^2}, (2\pi)^2\right) = F\left(T^2, \frac{A^2}{T^2}\right) - F\left(T_U^2, \frac{A^2}{T_U^2}\right)$$

For the energy density one obtains an exact value:

$$\rho = u^\mu u^\nu \langle \hat{T}^{\mu\nu} \rangle_{ren} = \frac{\pi^2}{30} T^4 \left(1 + \frac{5}{2\pi^2} \frac{A^2}{T^2}\right) - \frac{\pi^2}{30} T_U^4 \left(1 + \frac{5}{2\pi^2} (2\pi)^2\right)$$

for the *canonical stress-energy tensor*, there is a quantum-relativistic correction quadratic in the acceleration.

The exact value corresponds to the first term of the expansion in A^2 obtained in F.B., E. Grossi, Phys. Rev. D 92, 045037 (2015)

Conclusions

- Study of thermal equilibrium in QFT with acceleration

- The comoving observer – according to the Unruh effect – in the Minkowski vacuum – sees a thermal radiation. Thus, it is reasonable that, there is an absolute lower bound, for an accelerated fluid: $T < T_U$

- This conclusion applies to the *global equilibrium* and likely related to the Killing horizon, difficult to extend it to a general local thermodynamic equilibrium

For an interacting scalar field theory:

$$\langle 0_{\mathrm{M}}|\mathrm{T}[\widehat{\psi}(x),\ldots,\widehat{\psi}(x')]|0_{\mathrm{M}}\rangle = \frac{\mathrm{tr}(\exp[-2\pi\widehat{K}'/a]\mathrm{T}[\widehat{\psi}(x),\ldots,\widehat{\psi}(x')])}{\mathrm{tr}(\exp[-2\pi\widehat{K}'/a])}$$

Bisognano Wichmann 1975